PERIYAR UNIVERSITY

(NAAC 'A++' Grade with CGPA 3.61 (Cycle - 3) State University - NIRF Rank 56 - State Public University Rank 25)

SALEM - 636 011, Tamil Nadu, India.

CENTRE FOR DISTANCE AND ONLINE EDUCATION

(CDOE)

M.A.ECONOMICS

SEMESTER - II

CORE VI: MATHEMATICAL ECONOMICS

(Candidates admitted from 2025 onwards)

PERIYAR UNIVERSITY

CENTRE FOR DISTANCE AND ONLINE EDUCATION (CDOE)

M.A Economics 2025 admission onwards

CORE VI

Mathematical Economics

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SEMESTER—II

Core VI

MATHEMATICAL ECONOMICS

Unit I : Introduction to Linear Algebra

Sets-Basic concepts-Orderedsets-Relations-Orderrelations-MetricSpaces-openand closed sets– Convergence - Linear Algebra , Vectors, matrices, inverse, simultaneous linear equations, Cramer‘s rule for solving system of linear equations, input-output model, Hawkins - Simon condition, open and closed models , quadratic equation, characteristic (eigen) roots and vectors

Unit II : Differential Calculus

Introduction to Functions, Limits and Continuity, Derivatives –Concept of maxima& minima, elasticity and point of inflection. Profit & revenue maximization under perfect competition, under monopoly. Maximizing excise tax revenue in monopolistic competitive market, Minimization of cost etc.

Unit III : Optimization Techniques with Constraints

Functions of several variables, Partial and total, economic applications, implicit function theorem, higher order derivatives and Young‘s theorem, properties of linear homogenous functions, Euler's theorem, Cobb – Douglas Production Function-Constrained Optimization- Lagrangian Multiplier Technique- Vector and Matrix Differentiation -Jacobian and Hessian Matrices- Applications-Utility maximization, Profit maximization and Cost minimization.

Unit IV: Linear and Non-Linear Programming

Optimization with Inequality Constraints- Linear Programming–Formulation-Primal and Dual- Graphical and Simplex method-Duality Theorem-Non-Linear Programming-Kuhn-Tucker Conditions- Economic Applications.

Unit V: Economic Dynamics

Differential Equations-Basic Ideas-Types-Solution of Differential Equations (Homogenous and Exact)-Linear Differential Equations with Constant Coefficients (First and Second Order)-

Applications- Solow‘s Model- Harrod - Domar Model-Applications to Market models- Difference Equations - Types-Linear Difference Equations with Constant Coefficients (First and Second order) and solutions – Applications- Samuelson‘s Accelerator-Multiplier model- Cobweb model.

Textbooks:

1. GeoffRenshaw,(2016)MathsforEconomics,4EOxfordUniversityPress.
2. Mabbet A J(1986) Workout Mathematics for Economists, Macmillan Master Series, 4th Edition London.

References:

1. Carter,M.(2001). Foundations of Mathematical Economics,MITPress.
2. Chiang,A.C.and Wainwright,K.(2005).FundamentalMethodsofMathematical Economics, McGraw-Hill Education.
3. DowlingE. T., Mathematics for economists, Schaum Series (latest edition).

Web Resources

1. https://www.udemy.com/course/mathematics-for-economists-functions-and-derivatives/
2. https://www.classcentral.com/course/swayam-mathematical-economics-14187
3. https://www.coursera.org/learn/introduction-to-calculus

TABLE OF CONTENTS

UNIT 1

INTRODUCTION TO LINEAR ALGEBRA SETS

A collection of well- defined objects or Elements

Example

The set of even numbers less than 10 =>A= {2, 4, 6, 8}

Set Operations

Union (∪):

The union of two sets, A and B (denoted A ∪ B), contains all elements that are in either A or B, or both.

Example: If A = {1, 2, 3} and B = {3, 4, 5}, then A ∪ B = {1, 2, 3, 4, 5}.

Intersection (∩):

The intersection of two sets, A and B (denoted A ∩ B), contains only the elements that are common to both A and B.

Example: If A = {1, 2, 3} and B = {3, 4, 5}, then A ∩ B = {3}.

Difference (-):

The difference of two sets, A and B (denoted A - B), contains all elements that are in A but not in B.

Example: If A = {1, 2, 3} and B = {3, 4, 5}, then A - B = {1, 2}.

Complement ('):

The complement of a set A (denoted A') within a universal set U, contains all elements in U that are not in A.

Example: If U = {1, 2, 3, 4, 5} and A = {1, 2, 3}, then A' = {4, 5}.

Cartesian Product (×):

The Cartesian product of two sets A and B (denoted A × B) is the set of all possible ordered pairs where the first element comes from A and the second element comes from B.

Example: If A = {1, 2} and B = {a, b}, then A × B = {(1, a), (1, b), (2, a), (2, b)}.

Set Identities

1) Identity Law

$A \cup \emptyset = A, A \cap U = A$

2) Complement Law

$A \cup A^c = U, A \cap A^c = \emptyset$

3) Commutative Laws: $A \cup B = B \cup A, A \cap B = B \cap A$

4) Associative Laws

$A \cup (B \cup C) = (A \cup B) \cup C, A \cap (B \cap C) = (A \cap B) \cap C$

5) Distributive Laws

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C), A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

6) De Morgan's Law:

$(A \cup B)^c = A^c \cap B^c, (A \cap B)^c = A^c \cup B^c$

7) Absorption Laws

$A \cup (A \cap B) = A$, $A \cap (A \cup B) = A$

8) Complements of U and ∅

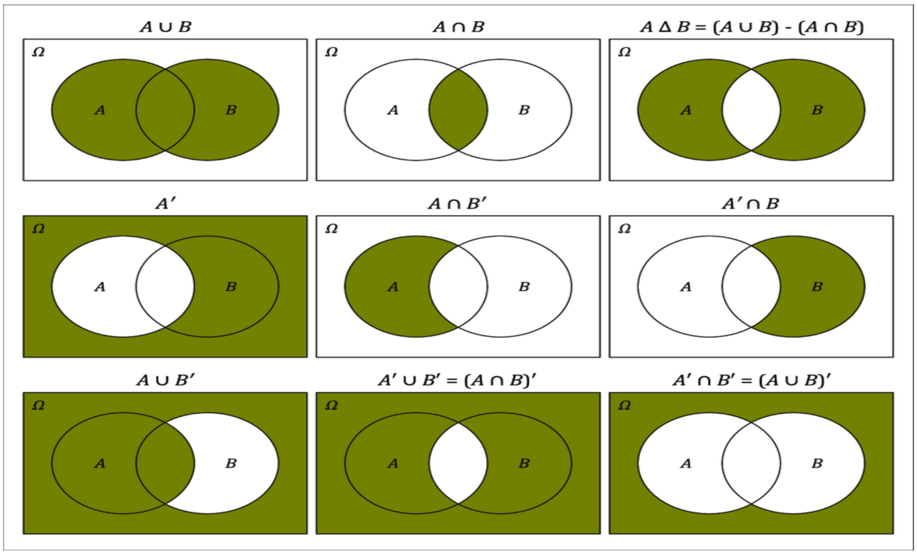
$U^c = \emptyset$, $\emptyset^c = U$

9) Set Difference Law

$A \setminus B = A \cap B^c$

Venn diagram

A Venn diagram is a diagram that helps us visualize the logical relationship between sets and their elements and helps us solve examples based on these sets. A Venn diagram typically uses intersecting and non-intersecting circles (although other closed figures like squares may be used) to denote the relationship between sets.



Set and Relation:

An ordered set consists of a set S and a relation ≤ on S.

Partial Order Properties:

1. Reflexive: For any element a in S, a ≤ a.

2. Antisymmetric: If a ≤ b and b ≤ a, then a = b.

3. Transitive: If a ≤ b and b ≤ c, then a ≤ c

Totally Ordered Set (Chain):

If every pair of elements in the set can be compared (either a ≤ b or b ≤ a), then the ordered set is called a totally ordered set or a chain.

Examples:

1. The natural numbers (N), integers (Z), rational numbers (Q), and real numbers (R)

   with their usual order are examples of ordered sets.

2. The set of all subsets of a set, ordered by inclusion, is another example.

Partially ordered set (poset)

A set with a partial order relation.

Totally ordered set (toset)

A set where every pair of elements is comparable.

Well-ordered set

A totally ordered set where every non-empty subset has a smallest element.

Relation:

A relation is a set of ordered pairs that connects elements from one or more sets.

For example, if you have sets A = {1, 2, 3} and B = {a, b}, a relation could be {(1, a), (2, b), (3, a)}.

Order Relation

An order relation is a special type of relation that establishes a method for comparing elements within a set.

Common examples include "less than or equal to" (≤) or "greater than" (>), which are used to order numbers.

Partial Order

A relation that is reflexive, antisymmetric, and transitive.

Total Order

A partial order where every pair of elements in the set is comparable.

Strict Partial Order

A relation that is irreflexive, asymmetric and transitive.

Examples:

1. The set of real numbers with the "less than or equal to" relation (≤) forms a partially ordered set.
2. The set of natural numbers with the "divides" relation is a partially ordered set.
3. The set of all subsets of a set, ordered by inclusion, is a partially ordered set.

Metric Space

A metric space is a pair (X, d), where: X is a non-empty set (the "points" in the space). d is a function called the metric (or distance function) that maps pairs of points in X to non-negative real numbers ($d: X \times X \rightarrow R^+$).

The metric d must satisfy the following properties for all x, y, z in X:

1. Non-negativity: $d(x, y) \geq 0$.
2. Identity: $d(x, y) = 0$ if and only if $x = y$.
3. Symmetry: $d(x, y) = d(y, x)$.
4. Triangle Inequality: $d(x, z) \leq d(x, y) + d(y, z)$.

Examples:

Euclidean Space:

The set of real numbers (R) with the usual distance function

$d(x, y) = |x - y|$ is a metric space.

Open Sets

A set is open if every point in the set has a neighbourhood that is entirely contained within the set.

Examples:

1. The open interval (0, 1) is an open set because for any point within (0, 1), you can find a small interval around it that is also entirely within (0, 1).
2. The set of all real numbers greater than 2, $(2, \infty)$, is an open set.
3. The empty set (∅) is both open and closed.

Not Open

The closed interval is not open because the endpoint 0 and 1 do not have a neighbourhood entirely within the set.

The set [0, 1) is neither open nor closed.

Closed Sets:

A set is closed if it contains all its boundary points.

Examples:

1. The closed interval is a closed set because it contains all its boundary points, 0 and 1.
2. The set of all real numbers less than or equal to 2, (-∞, 2], is a closed set.
3. The set {1, 2, 3} is a closed set.

Not Closed

The open interval (0, 1) is not closed because it does not contain its boundary points, 0 and 1.

The set [0, 1) is neither open nor closed.

Neither Open Nor Closed

Examples

1. The set [0, 1) is neither open nor closed because it contains the boundary point 0, but not 1.

2. The set (0, 1] is neither open nor closed because it contains the boundary point 1, but not 0.

LINEAR ALGEBRA

Linear algebra is the branch of mathematics focused on the study of vector spaces, linear transformations, and matrices. It deals with linear combinations, systems of linear equations, and their properties.

Vector Spaces

These are collections of objects (vectors) that can be added together and multiplied by scalars (numbers) while preserving certain properties (like associativity and commutativity).

Matrix

A matrix is a rectangular array of elements (numbers, symbols, or expressions) organized into rows and columns.

Elements

The individual numbers, symbols, or expressions within the matrix are called its elements or entries.

Rows and Columns

Matrices are characterized by the number of rows (horizontal arrangements) and columns (vertical arrangements) they have.

Notation

Matrices are typically denoted by uppercase letters (e.g., A, B, C).

Size or Order

The size or order of a matrix is determined by the number of rows and columns it contains, usually written as "m x n" (m rows and n columns).

Examples:

A 2x3 matrix (2 rows, 3 columns): $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$

Based on Shape and Size

Row Matrix

A matrix with only one row.

Column Matrix

A matrix with only one column.

Square Matrix

A matrix where the number of rows equals the number of columns.

Rectangular Matrix

A matrix where the number of rows and columns are not equal.

Horizontal Matrix

A matrix where the number of columns is greater than the number of rows.

Vertical Matrix

A matrix where the number of rows is greater than the number of columns.

Based on Elements:

Diagonal Matrix

A square matrix where all non-diagonal elements are zero.

Scalar Matrix

A diagonal matrix where all diagonal elements are equal.

Identity Matrix

A diagonal matrix with ones on the diagonal and zeros elsewhere.

Zero Matrix (Null Matrix)

A matrix where all elements are zero.

Triangular Matrix

A square matrix where all elements above or below the main diagonal are zero.

Upper Triangular Matrix

A square matrix where all elements below the main diagonal are zero.

Lower Triangular Matrix

A square matrix where all elements above the main diagonal are zero.

Symmetric Matrix

A square matrix where the elements are mirrored across the main diagonal

Skew-Symmetric Matrix

A square matrix where the elements are mirrored across the main diagonal with a negative sign

Other Important Types:

Singular Matrix

A square matrix whose determinant is zero (and therefore does not have an inverse).

Non-Singular Matrix

A square matrix whose determinant is non-zero (and therefore has an inverse).

Orthogonal Matrix

A square matrix whose inverse is equal to its transpose.

Matrix Inverse

If A is a non-singular square matrix, there is an existence of n x n matrix A-1, which is called the inverse matrix of A such that it satisfies the property:

1. $AA^{-1} = A^{-1}A = I$, where I is the Identity matrix
2. The identity matrix for the 2 x 2 matrix is given by $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
3. The inverse matrix is $A^{-1} = \frac{1}{|A|} adj\ A$

Find A^{-1} if $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{pmatrix}$

Solution

Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{pmatrix}$

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{vmatrix} = 1 \neq 0$$

We know that $A^{-1} = \frac{1}{|A|} adj\ A$

Find the cofactor matrix of

C11 = (1·0−4·6)= -24, C12 = −(0·0−4·5) = 20, C13 = (0·6−1·5) = -5,

C21 = −(2·0−3·6) = 18, C22 = (1·0−3·5) = -15, C23 = −(1·6−2·5) = 4 ,

C31 = (2·4−3·1) = 5, C32 = −(1·4−3·0) = -4, C33 = (1·1−2·0) = 1

Cofactor matrix of A = $\begin{pmatrix} -24 & 20 & -5 \\ 18 & -15 & 4 \\ 5 & -4 & 1 \end{pmatrix}$

$$Adj\ A = C^T = \begin{pmatrix} -24 & 18 & 5 \\ 20 & -15 & -4 \\ -5 & 4 & 1 \end{pmatrix}$$

$$A^{-1} = \frac{1}{|A|} adj\ A$$

$$= \frac{1}{1} \begin{pmatrix} -24 & 18 & 5 \\ 20 & -15 & -4 \end{pmatrix}$$

$-5 \quad 4 \quad 1$

$$A^{-1} = \begin{pmatrix} -24 & 18 & 5 \\ 20 & -15 & -4 \\ -5 & 4 & 1 \end{pmatrix}$$

SIMULTANEOUS LINEAR EQUATIONS

Linear equations are the equations in which the highest power of the variable is one. The result of the linear equation is always a straight line. Thus, simultaneous linear equations are the system of two linear equations in two or three variables that are solved together to find a common solution

Methods to Solve Simultaneous Linear Equations

Generally, three different methods are used to solve the simultaneous linear equations.

They are:

- Substitution Method
- Elimination Method
- Graphical Method

Substitution Method

Follow the below steps to solve the system of linear equations or simultaneous linear equations using the substitution method:

1. Rearrange one of the given equations to express x in terms of y.

2. Now, the expression for x can be substituted in the other equation to find the value

of y.

3. Finally, substitute the value of y in any of the equations to find the value of x.

Example 1:

Solve the following system of linear equations using the substitution method:

$3x - 4y = 0$, $9x - 8y = 12$

Solution:

Given:

$3x - 4y = 0$ …(1) & $9x - 8y = 12$ …(2)

Step 1: Rearrange equation (1) to express x in terms of y:

$\Rightarrow 3x = 4y$

$\Rightarrow x = 4y/3$ …(3)

Now substitute $x = 4y/3$ in (2), we get

$\Rightarrow 9(4y/3) - 8y = 12$

$\Rightarrow (36y/3) - 8y = 12$

$\Rightarrow 12y - 8y = 12$

$\Rightarrow 4y = 12$

$\Rightarrow y = 12/4 = 3$

Hence, the value of y is 3

Now, substitute y= 3 in (3) to get the value of x

$\Rightarrow x = [4(3)]/3 = 12/3 = 4$

Therefore, x = 4

Thus, the solution for the simultaneous linear equations (x, y) is (4, 3).

Elimination Method

To find the solutions (x, y) for the system of linear equations using the elimination method, go through the below steps:

1. Multiply the given equations by a constant, so as to make the coefficients of the variables in the equations equal.
2. Add or subtract the equations to eliminate the variable having the same coefficients.
3. Now, solve the equation for one variable.
4. Substitute the variable value in any of the equations to find the value of the other variable.

5 .Finally, the ordered pair (x, y) is the solution of the simultaneous equation.

Example 2:

Solve the system of equations using the elimination method:

$2x+3y= 11$, $x+2y = 7$

Solution:

Given: $2x+3y= 11$ ...(1)

$x+2y = 7$ ... (2)

Now, multiply the equation (2) by 2, we get

$2x + 4y = 14$ ...(3)

Now, solve the equation (1) and (3),

$2x + 3y = 11$

$2x + 4y = 14$

– – –

$0 – y = -3$

$\Rightarrow -y = -3$

$\Rightarrow y = 3$

Now, substitute $y = 3$ in equation (2),

$x + 2(3) = 7$

$x + 6 = 7$

$x = 1$

Hence, $x = 1$ and $y = 3$

Therefore, the solution for the system of equations 2x+3y= 11and x+2y = 7 is x = 1 and y=3.

Graphical Method

`To solve the simultaneous linear equations graphically, follow the below steps:

1. First, find the coordinates of two equations simultaneously by substituting the values of x as 1, 2, 3, 4, etc.
2. Now, plot the points and we get two straight lines.
3. Observe the common point of intersection of two straight lines.
4. The common point of intersection of two straight lines is the solution of the simultaneous linear equation.

Example 3:

Solve the simultaneous linear equations graphically: x+y = 4 and x -y = 0.

Solution:

Given:

x+y = 4 …(1)

x -y = 0 …(2)

Let us take equation (1), x+y = 4

When x = 1,

⇒ y = 4-1 = 3

When x = 2,

⇒y= 4-2 = 2

When x = 3,

⇒ y = 4-3 = 1

X	1	2	3
Y	3	2	1

Now, take the second equation x-y = 0

When x = 1,

⇒ -y = 0-1

⇒ y = 1

When x = 2,

⇒ -y = 0-2

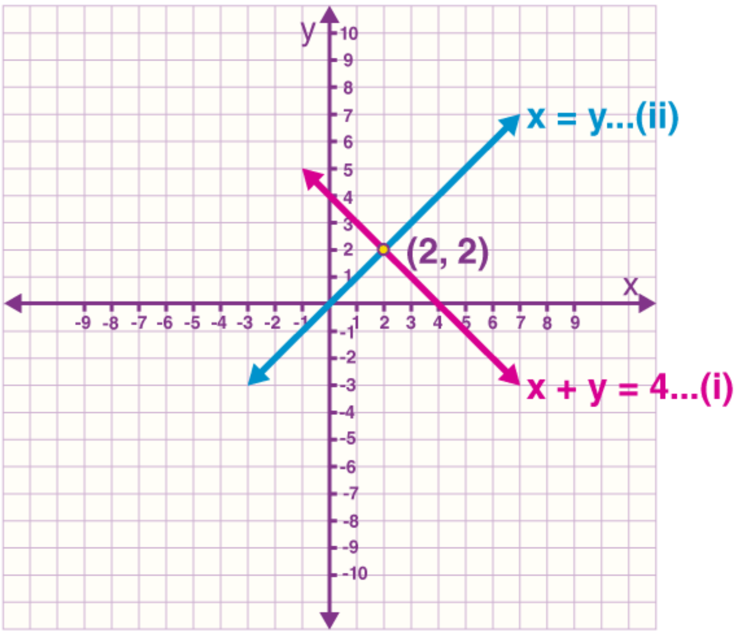
⇒ y = 2

When x = 3,

⇒ -y = 0-3

⇒ y = 3

X	1	2	3
Y	1	2	3



Cramer's Rule

Cramer's rule is one of the important methods applied to solve a system of equations. In this method, the values of the variables in the system are to be calculated using the determinants of matrices. Thus, Cramer's rule is also known as the determinant method.

Example Solve the following system of equations using Cramer's rule:

2x – y = 5, x + y = 4

Solution:

Given,

2x – y = 5 & x + y = 4

Let us write these equations in the form AX = B.

$$\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$$

Here

$$A = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \end{pmatrix}, \quad B = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$$

Now

$$D = |A| = \begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} = (2 + 1) = 3 \neq 0$$

So, the given system of equations has a unique solution.

$5 \quad -1 \quad (\quad)$

$$D_x = \left| \begin{matrix} & \\ 4 & 1 \end{matrix} \right| = 5 + 4 = 9 \neq 0$$

$$D_y = \begin{vmatrix} 2 & 5 \\ 1 & 4 \end{vmatrix} = (8-5) = 3 \neq 0$$

Therefore,

x = D_x/D = 9/3 = 3

y = D_y/D = 3/3 = 1

Solve the following system of equations using Cramer's rule:

x + y + z =6 , y + 3z = 11, x + z =2y

Solution:

Given,

x + y + z =6 , y + 3z = 11, x – 2y + z = 0

Let us write these equations in the form AX = B.

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 1 & -2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 11 \\ 0 \end{pmatrix}$$

$$D = |A| = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 1 & -2 & 1 \end{vmatrix} = 9 \neq 0$$

$$D_x = \begin{vmatrix} 6 & 1 & 1 \\ 11 & 1 & 3 \\ 0 & -2 & 1 \end{vmatrix} = 9 \neq 0 \quad D_y = \begin{vmatrix} 1 & 6 & 1 \\ 0 & 11 & 3 \\ 1 & 0 & 1 \end{vmatrix} = 18 \neq 0 \quad D_z = \begin{vmatrix} 1 & 1 & 6 \\ 0 & 1 & 11 \\ 1 & -2 & 0 \end{vmatrix} = 27 \neq 0$$

Thus,

$x = D_x/D = 9/9 = 1$, $y = D_y/D = 18/9 = 2$, $z = D_z/D = 27/9 = 3$

Input-Output Model:

This model analyzes the interdependence between different sectors of an economy, showing how the output of one sector is used as input by other sectors for production.

Hawkins-Simon Condition:

The condition for the nxn matrix of (I−A) to have an inverse of nonnegative elements is that its principal leading minors be positive. This is known as the Hawkins-Simon conditions.

Matrix (I-A):

- I: Represents the identity matrix, a square matrix with ones on the main diagonal and zeros elsewhere.
- A: Is the input-output or technology matrix, where each element represents the amount of input from sector i required to produce one unit of output in sector j.
- (I-A): The difference between the identity matrix and the technology matrix.

Example

Let's consider a simple 2x2 input-output matrix A:

$$A = \begin{pmatrix} 0.4 & 0.2 \\ 0.3 & 0.5 \end{pmatrix}$$

Calculate (I - A)

Let $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$I - A = \begin{pmatrix} 0.6 & -0.2 \\ -0.3 & 0.5 \end{pmatrix}$$

1. **Calculate Principal Leading Minors:**
   - First Minor: The determinant of the 1x1 matrix formed by the first element: |0.6| = 0.6 (Positive)
   - Second Minor: The determinant of the 2x2 matrix (I - A): (0.6 * 0.5) - (-0.2 * -0.3) = 0.3 - 0.06 = 0.24 (Positive)
2. **Conclusion:**

   Since both principal leading minors are positive, the Hawkins-Simon condition is satisfied, and the economy is considered viable.

The Open Model:

If, besides the n industries, the model contains an "open" sector (say, households) which exogenously determines a final demand (non-input demand) for the product of each industry and which supplies a primary input (say, labour service) not produced by the n industries themselves, then the model is an open one

The Closed Model

If the exogenous sector of the open input-output model is absorbed into the system as just another industry, the model will become a closed one. In such a model, final demand and primary input do not appear; in their place will be the input requirements and the output of the newly conceived industry. All goods will now be intermediate in nature, because everything is produced only for the sake of satisfying the input requirements of the (n + 1) sectors in the model.

Characteristic equation:

Let A be any square matrix of order n then the characteristic equation of A is $|A - \lambda I| = 0$ where λ is scalar and I is the unit matrix.

Eigen values and Eigen vectors of a Real matrix:

Eigenvalues or Characteristic roots:

Let $A = [a_{ij}]$ be a square matrix. The characteristic equation of A is $|A - \lambda I| = 0$.

The roots of the characteristic equation are called Eigenvalues of A.

Eigenvector or Latent vector:

Let $A = [a_{ij}]$ be a square matrix. If there exists a non zero vector $X = \begin{pmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{pmatrix}$

such that $AX = \lambda X$, then vector X is called an Eigenvector of A corresponding to the Eigenvalue of λ.

Properties of Eigenvalues and Eigenvectors:

Property

i) Sum of eigenvalues = Sum of diagonal elements.

ii) Product of eigenvalues = Value of the determinant

iii) Every square matrix and its transpose have the same eigenvalues.

iv) If $\lambda_1, \lambda_2, ..., \lambda_n$ are the eigenvalues of matrix A then $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, ..., \frac{1}{\lambda_n}$ are the eigenvalues of matrix A^{-1}.

v) If $\lambda_1, \lambda_2, ..., \lambda_n$ are the eigenvalues of matrix A, then the matrix A^m has the eigen values $\lambda_1^m, \lambda_2^m, ..., \lambda_n^m$

If $\lambda_1, \lambda_2, ..., \lambda_n$ are the eigenvalues of matrix A, then the matrix kA has the eigen values $k\lambda_1, k\lambda_2, ..., k\lambda_n$

.

vi) The eigenvectors corresponding to distinct eigenvalues of a real symmetric matrix are orthogonal.

vii) If a real symmetric matrix of order 2 has equal eigenvalues then the matrix is a scalar matrix.

viii) The eigenvectors of a matrix A is not unique.

ix) If $\lambda_1, \lambda_2, ..., \lambda_n$ are the distinct eigenvalues of matrix A, then the corresponding eigenvectors $X_1, X_2, ..., X_n$ form a linearly independent set.

x) If two or more eigenvalues are equal it may or may not be possible to get linearly independent eigenvectors corresponding to the equal roots.

1. **Obtain the characteristic equation of $\begin{pmatrix} 1 & -2 \\ -5 & 4 \end{pmatrix}$.**

Solution:

$\begin{pmatrix} 1 & -2 \end{pmatrix}$

Let A = $\begin{pmatrix} -5 & 4 \end{pmatrix}$

The characteristic equation of A is $\lambda^2 - c_1\lambda + c_2 = 0$

$$c_1 = sum of\ the main diagonal\ elements$$
$$= 1 + 4 = 5$$

$$c_2 = |A|$$
$$= \begin{vmatrix} 1 & -2 \\ -5 & 4 \end{vmatrix}$$
$$= 4 - 10$$
$$= -6$$

$$Hence the characteristic equation is$$
$$\lambda^2 - (5)\lambda + (-6) = 0$$
$$\lambda^2 - 5\lambda - 6 = 0$$

2. Find the sum and product of the eigen values of the matrix $\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$**.**

Solution:

$$sum of\ the eigen values = sum of the diagonal\ elements$$
$$= (-1) + (-1) + (-1)$$
$$= -3$$

$$product\ of\ the eigen values = \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$$= -1(1-1) - 1(-1-1) + 1(1+1)$$
$$= -1(0) - 1(-2) + 1(2)$$
$$= 4$$

3. The product of the eigenvalues of the matrix $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ **is 16, Find the third eigenvalue.**

Solution:

let the eigenvalues of the matrix A be $\lambda_1, \lambda_2, \lambda_3$.

Given $\lambda_1\lambda_2 = 16$

we know that $\lambda_1\lambda_2\lambda_3 = |A|$

$$= \begin{vmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{vmatrix}$$

$$= 6(9-1) + 2(-6+2) + 2(2-6)$$
$$= 6(8) + 2(-4) + 2(-4)$$

$$= 32$$
$$16\lambda_3 = 32$$
$$\lambda_3 = 2$$

4. Find the eigen values and eigen vectors of the matrix $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

Solution:

The characteristic equation is $|A - \lambda I| = 0.$

ie., $\begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & 0-\lambda \end{vmatrix} = 0$

ie., $(-2-\lambda)\left[-\lambda(1-\lambda)-12\right] - 2[-2\lambda - 6] - 3[-4+1-\lambda] = 0$

ie., $(-2-\lambda)\left[\lambda^2 - \lambda - 12\right] + 4\lambda + 12 + 9 + 3\lambda = 0$

ie., $\lambda^3 + \lambda^2 - 21\lambda - 45 = 0$ (1)

Now, $(-3)^3+(-3)^2-21(-3)-45=-27+9+63-45=0$

$\therefore -3$ is a root of equation (1)

Dividing $\lambda^3+\lambda^2-21\lambda-45$ *by* $\lambda+3$

$$\begin{array}{r|rrrr} -3 & 1 & 1 & -21 & -45 \\ & 0 & -3 & 6 & 45 \\ \hline & 1 & -2 & -15 & 0 \end{array}$$

Remaining roots are given by

$\lambda^2-2\lambda-15=0$

ie., $(\lambda+3)(\lambda-5)=0$

ie., $\lambda=-3, 5.$

$\therefore$ The eigen values are -3, -3, 5

The eigen vectors of A are given by $\begin{bmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Case 1: $\lambda=-3$

Now $\begin{bmatrix} 2+3 & 2 & -3 \\ 2 & 1+3 & -6 \\ -1 & -2 & 3 \end{bmatrix}=\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix}$

$\therefore x_1+2x_2-3x_3=0$

Put $x_2=k_1, x_3=k_2$

Then $x_1=3k_2-2k_1$

$\therefore$ The general eigen vectors corresponding to $\lambda=-3$ is $\begin{bmatrix} 3k_2-2k_1 \\ k_1 \\ \end{bmatrix}$

$\left|\lfloor \quad k_2 \quad \rfloor\right|$

When $k_1 = 0, k_2 = 1$, We get the eigen vector $\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$

When $k_1 = 1, k_2 = 0$, We get the eigen vector $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$

Hence the two eigen vectors corresponding to $\lambda = -3$ are $\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$.

These two eigen vectors corresponding to $\lambda = -3$ are linearly independent.

Case 2: $\lambda = 5$

$$\begin{bmatrix} -2-5 & 2 & -3 \\ 2 & 1-5 & -6 \\ -1 & -2 & -5 \end{bmatrix} = \begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix} = \begin{bmatrix} -1 & -2 & -5 \\ 0 & -8 & -16 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore -x_1 - 2x_2 - 5x_3 = 0$

$-8x_2 - 16x_3 = 0$

A solution is $x_3 = 1, x_2 = -2, x_1 = -1$

$\therefore$ Eigen vector corresponding to $\lambda = 5$ is $\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$.

5. Find the eigen vector of the matrix A= $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ -4 & 4 & 3 \end{pmatrix}$

Solution:

The characteristic equation of A is $\begin{vmatrix} 1-\lambda & 1 & 1 \\ 0 & 2-\lambda & 1 \\ \end{vmatrix} = 0$

$$\begin{pmatrix} -4 & 4 & 3-\lambda \end{pmatrix}$$

$$(1-\lambda)[(2-\lambda)(3-\lambda)-4]-1[0+4]+1[0+4(2-\lambda)]=0$$
$$(1-\lambda)\left(\lambda^2-5\lambda+6-4\right)-4+8-4\lambda=0$$

$$(1-\lambda)\left(\lambda^2-5\lambda+2\right)+4-4\lambda=0$$
$$(1-\lambda)\left(\lambda^2-5\lambda+2+4\right)=0$$
$$(1-\lambda)\left(\lambda^2-5\lambda+6\right)=0$$
$$(1-\lambda)(\lambda-2)(\lambda-3)=0$$

$\therefore$ The eigen values of A are $\lambda = 1, 2, 3.$

The eigen vectors are given by $\begin{pmatrix} 1-\lambda & 1 & 1 \\ 0 & 2-\lambda & 1 \\ -4 & 4 & 3-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

Case 1 $\lambda = 1$

$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ -4 & 4 & 2 \end{pmatrix} \sim \begin{pmatrix} -4 & 4 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$-4x_1+4x_2+2x_3=0$$
$$x_2+x_3=0$$

A solution is $x_3 = 2, x_2 = -2, x_1 = -1$

$\therefore$ Eigen vector $X_1 = \begin{pmatrix} -1 \\ -2 \\ 2 \end{pmatrix}$

Case 2 $\lambda = 2$

$$\begin{pmatrix} -1 & 1 & 1 \end{pmatrix} \quad \begin{pmatrix} -1 & 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 1 \\ -4 & 4 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$-x_1 + x_2 + x_3 = 0$$
$$x_3 = 0$$

A solution is, $x_3 = 0, x_2 = 1, x_1 = 1$

$\therefore$ Eigen vector $X_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

Case 3 $\lambda = 3$

$$\begin{bmatrix} -2 & 1 & 1 \\ 0 & -1 & 1 \\ -4 & 4 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$-2x_1 + x_2 + x_3 = 0$$
$$-x_2 + x_3 = 0$$

A solution is, $x_3 = 1, x_2 = 1, x_1 = 1$

$\therefore$ Eigen vector $X_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

UNIT II

DIFFERENTIAL CALCULUS

LIMIT AND CONTINUTITY

Introduction

Although this chapter contains very little actual mathematics, it is probably the most important in the book. It aims at introducing three of the difficulties which beset the newcomer to mathematics, but most of the difficulties each is quite simple and easy to grasp arise out of a failure to appreciate the other of these points. Consequently this chapter should be read very carefully, and no attempt should be made to hurry to later chapters.

Continuity

It is impossible for any car to change its speed suddenly from 30 m.p.h. to 60 m.p.h. However fast it may be accelerating, at some time or the other before reaching 60 m.p.h. it must be traveling (if only for a moment) at 50 m.p.h. At some time it must be doing 40 m.p.h. In between these moments it will show meter needle registers speeds between 30 m.p.h. and 60 m.p.h. At one moment it must be 38.8

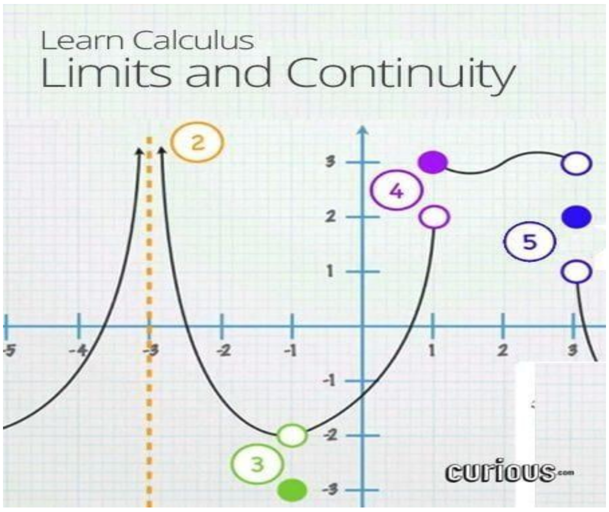
m.p.h. And at some time between these moments it will show 38.8 m.p.h., and at another time 38.9 m.p.h.

On the other hand, it is impossible for the retail price of a pound of sugar to change suddenly from 7p. to 10p. It is true that it need not change from 7p. to 8p., but suddenly for a pound of a halfpenny. But although the price per pound may be 7p. one day and 7½p. another, and conceivably 7¼p. at some time in between (by a sales arrangement of 7p. for this pound and 7½p. for the next), it is quite inconceivable that there is any one moment when you can go into a shop and buy one pound of sugar for 7⅜p.

Now we have the speed of a car change; but the prices of sugar change: they are variable quantities. Bfrom 30 m.p.h. to 60 m.p.h. by passing through all the intermediate speeds. The retail price of sugar varies discontinuously, changing from 7p. to 7½p. abruptly.

It is possible to illustrate these ideas graphically as in Diagram 7.1. With the car, we may choose any speed at all between 30 m.p.h. and 60 m.p.h. and determine from the diagram the time at which the car was traveling at that speed. For example, it was traveling at 43.3 m.p.h. after 6½ minutes.

But if in Diagram 7.2 we choose a price of 8½p. or of 10½p., we see that at no time did that price operate. The jumps from 8p. to 9p. and from 9p. to 11p. were abrupt ones; they were discontinuous.



Since both of these variables have values which depend on time, we say that they are functions of time. The speed is a continuous function of time; the price is a discontinuous function of time.

3. Limits

Let 'x' be a real variable and 'b' a finite real number. We often say "x → a" (x tends to a). This "x → a" is associated with the approach of the successive values of the variable x towards a given number a.

Limiting value of a function

What value does one variable 'y' approach when another variable 'x' approaches a specific finite quantity 'a'? This question makes proper sense only if y is a function of x.

Suppose, y = f(x). Here, our problem is to what value y (i.e. f(x)) approaches when x approaches 'a'. We describe this situation by writing:

$\lim_{x \to a} f(x) = l$ or f(x)=l when x=a

Eating an apple

We have already had an informal introduction to the idea of the limit in Chapter II, where we considered the sum of an infinite series. It is now time to consider the matter a little more formally, although we shall once again begin by referring to the problem of eating an apple. The problem, we may remember, is the practical one of how to consume the whole of an apple if each bite bites off one half of what is left after the previous bite. We may tabulate the process thus:

After 1 bite, we have eaten 1/2 of the apple; 1/2 remains.

After 2 bites, we have eaten 1/2 + 1/4 of the apple; 1/4 remains.

After 3 bites, we have eaten 1/2 + 1/4 + 1/8 of the apple; 1/8 remains.

After 4 bites, we have eaten 1/2 + 1/4 + 1/8 + 1/16 of the apple; 1/16 remains.

And so on.

No matter how many bites we take, there will always remain a portion equal in size to the last bite. By taking enough bites we may make this remaining portion as small as we please; but there will never be nothing left. After one bite, 0.5 of the apple remains. After two bites 0.25 remains. After five bites 0.03125 remains. After ten bites as little as 0.0009775 remains, while after twenty bites there is only 0.000000095. But always something remains. We can get as close to zero as we wish; but never quite reach it. We say, in fact, that zero is the Limit to which the amount of apple remaining tends as the number of bites become indefinitely great.

The sum of an infinite series

Let us now consider the G.P.

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots,$$

We know from Chapter II that the sum of terms of this series is given by

Now as the number of terms increases, this last expression becomes closer and closer to 2, because the part of it arising from becomes closer and closer to zero. On the other hand, it is never exactly zero, and so the sum of the series never quite reaches 2; but if we take enough terms we can get as close to 2 as we wish.

Because of this, we say that 2 is the limit to which the sum of the G.P. tends as the number of terms increases indefinitely. Furthermore, we say that 2 is the sum of the infinite series. This is a distinction which is most important, and which will become clearer as we proceed. The point to notice is that the sum of a finite series can be obtained quite simply by adding up the terms. As the number of terms increases, so the sums alter. If the series is convergent, the successive sums will tend to a limit. We define the sum of an infinite (convergent) series to be this limit.

Maxima and Minima

A. Maxima

A function f(x) is said to have attained its 'Maximum value' or 'Maxima' at x = a, if the function stops to increase and begins to decrease at x = a. In other words, $f(x_1)$ is a maximum value of a function 'f', if it is the highest of all its values for values of x in some neighbourhood of A (Fig. 3.1).

B. Minima

A function f(x) is said to have attained its 'Minimum value' or 'Minima' at x = b, if the function stops to decrease and begins to increase at x = b. In other words, $f(x_2)$ is a minimum value of a function 'f' if it is the lowest of all its values for values of x in some neighbourhood of B (Fig. 3.1).

The Maxima and Minima of the function are called the 'Extreme Values' of the function.

Maxima and Minima of One Variable

Let us consider a function Y = f(x). If we plot this function, the function takes the form as given in the Figure 3.1. We consider three points A, B, and C, where dy/dx = 0 in each case. That is, in all stationary levels, the derivative is zero.

(1) At Point ‘A’

We call the point 'A' a maximum point because Y has a maximum value at this point when X = OX_1. Therefore, we say that at point A, Y is maximum. It means that the value of Y at A is higher than any value on either side of A, and also on the left side of point 'A', the curve is increasing and decreasing on the right side of 'A'. This also means that the value of Y increases with the increase in X up to the point A. But it must fall afterward.

Conditions for Maxima and Minima

(1) At Point 'A'

Point A has been reached. As the value of X increases from X_1 to X_2, the slope of the curve is decreasing or the slope of the curve changes from zero to negative values (since dy/dx = 0 at point 'A'). Thus, if 'A' is to be maximum,

i) dy/dx = 0 and also

ii) $d^2y/dx^2 < 0$ i.e., -ve at point 'A'.

(2) At Point 'B'

We call the point 'B' a minimum point because Y has a minimum value when X = OX_2. Therefore, we say that at point 'B', Y is minimum. It means that the value of Y at 'B' is lower than any value on either side of B and also on the left side of point 'B', the curve is decreasing and increasing on the right side of 'B'. This also means that the value of Y decreases with the increase in X up to the point 'B'. But, it must increase after point 'B' has been reached. As the value of X increases from X_2 to X_3, the slope of the curve is increasing or the slope of the curve changes from zero to positive values (since dy/dx = 0 at point 'B').Thus if 'B' is to be minimum,

i) dy/dx = 0 and also

ii) $d^2y/dx^2 > 0$ i.e., +ve at point 'B'.

Table 3.2: Conditions for Maxima and Minima

	Maxima	Minima
1. First Order Condition (Necessary Condition)	f'(x) or dy/dx = 0	f'(x) or dy/dx = 0
2. Second Order Condition (Sufficient Condition)	f''(x) or $d^2y/dx^2 < 0$ or -ve	f''(x) or $d^2y/dx^2 > 0$ or +ve

(3) At Point 'C'

Point of Inflexion is a point at which a curve is changing from concave upward to concave downward, or vice versa. In the Figure 3.1, we call this point 'C' a 'Point of Inflection' or 'Inflexional Point'. Because on either sides of point 'C', the curve slopes upwards. Therefore, on either sides of 'C', the first order derivative is greater than zero i.e., positive except at point 'C'. Hence, this point is called 'Inflexional Point' or 'Point of Inflection' because of mere bend in the curve. At the inflexional points, the second order derivative is equal to zero.

Inflectional points may be stationary and inflectional. In this case, both first and second order derivatives are zero. If the point is inflexional and non stationary, the first order derivative is not equal to zero. But the second order derivative is equal to zero.

Thus, if 'C' is to be "Inflexional point", then i) $dy/dx \geq 0$ and also ii) $d^2y/dx^2 = 0$.

Examples:

1. Given the function $y = x^3 - 3x^2 + 7$, find the point of inflexion.

Solution:

$dy/dx = 3x^2 - 6x$

The first condition for inflexion is $dy/dx = 0$

Therefore, $3x^2 - 6x = 0$

$x(3x - 6) = 0$

i) $x = 0$ or ii) $3x - 6 = 0$

$3x = 6$

$x = 6/3 = 2.$

Thus, $dy/dx = 0$, when $x = 0$ or $x = 2$.

The second condition for inflexion is $d^2y/dx^2 = 0$

$d^2y/dx^2 = 6x - 6$

$6x - 6 = 0$

$6x = 6$

$x = 6/6 = 1 > 0.$

The point of inflexion is $x = 0$ and $y = 7$ or $x = 2$ and $y = 3$.

Find the maxima and minima of the function $y = 2x^3 - 6x$.

Solution:

$y = 2x^3 - 6x$

$dy/dx = 6x^2 - 6$

At the maximum or minimum $dy/dx = 0$

Therefore, $6x^2 - 6 = 0$

$6x^2 = 6$

$x^2 = 6/6$

$x^2 = 1$

$x = \pm 1$

Derivatives

So far we concerned ourselves with functions of one independent vari-

$x = -1$ and $x = 1$ give maximum or minimum

$dy/dx = 6x^2 - 6x$

$d^2y/dx^2 = 12x - 6$

When $x = -1$,

$d^2y/dx^2 = -18 < 0$ i.e., negative

When $x = 1$,

$d^2y/dx^2 = 6 > 0$ i.e., positive.

Therefore, $x = -1$ gives the maximum value of the function and $x = 1$ gives the minimum value of the function.

The arc elasticity of demand

We now move on to consider the elasticity of demand. The analysis of the elasticity of demand is closely analogous to that of the elasticity of supply. Therefore we can move a little faster now that the basic principles have been established.

The basic features of the market demand function were examined in sections 4.13 and 8.8. The demand function is the functional relationship between the quantity of a good which buyers, in aggregate, demand (that is, wish to buy) and the market price. We can write the demand functionas

$$q=g(p)$$

where q, the dependent variable, denotes the quantity demanded (measured in some physical unit) and p denotes the purchase price of each unit. We have written g(p) to denote the demand function so as to avoid any danger of confusing it with the supply function q = f(p) examined earlier in this chapter. Economic theory suggests that the function g(p) is normally negatively sloped, because buyers will wish to buy smaller quantities at higher prices. If there is only one (monopoly) supplier of the good, the market demand function is also the demand function for that firm's product. However, under perfect competition a typical firm can sell as much output as it wishes at the ruling market price, and thus faces a horizontal inverse demand function.

As with the supply function, we assume that the demand function is smooth and continuous. For more detail on the demand function, see any introductory micro economics text book.

Given the market demand function, introductory economics textbooks define the arc elasticity of demand, which we will denote by E^D_A, as:

$$E^D_A = \text{(percentage change in quantity demanded)} / \text{(percentage change in price)}$$

Profit maximization

In sections examined the firm's total, average and marginal cost functions and their interrelationships. we looked at the market demand curve for a good and introduced the concepts of total and marginal revenue. Now we can bring cost and demand conditions together and thus examine how a firm should proceed if it wishes to achieve maximum profit. Because we have already done most of the hard work earlier in this chapter, this turns out to be less difficult than you might expect.

As usual we shall develop the analysis by means of a worked example, and the generalize out findings. We will consider first the case of monopoly, then perfect competition.

Profit maximization with monopoly

EXAMPLE . 1

Suppose there is a monopolist firm with total cost function $TC=q^2 + 2q + 500$, and the market demand function for the product is $q=-0.5p+ 100$.

Given that the firm is a monopolist-that is, it is the only supplier of the product in question-the market demand function is also the demand function for the monopoly firm's product, as noted earlier.

If the monopolist chooses to fix the price at which it offers the product for sale, then the demand function determines the quantity that consumers will buy at that price. For example, if the price is fixed at $p= 20$, then q is determined by the demand function as $q=-0.5p+100=90$, Alternatively, if the monopolist chooses to produce a certain quantity and offer it for sale, then the demand function determines the price that consumers are willing to pay for that quantity.

For example, if the monopolist produces and sells q = 80, then in the demand function we have 80=-0.5p + 100, and the solution to this equation is p = 40. Thus the monopolist can fix either the price or the quantity sold, but not both.

To obtain the firm's total revenue function we first rearrange the demand function above to derive the inverse demand function, which is p=-2q+200.

Given the definition of total revenue as TR = pq, we then use the inverse demand function to eliminate p from the TR expression. This gives

$$TR=pq=(-2q+200)q=-2q^2+200q$$

In figure 8.14(a) we have graphed the total cost and total revenue functions. In this example they are both quadratic functions.

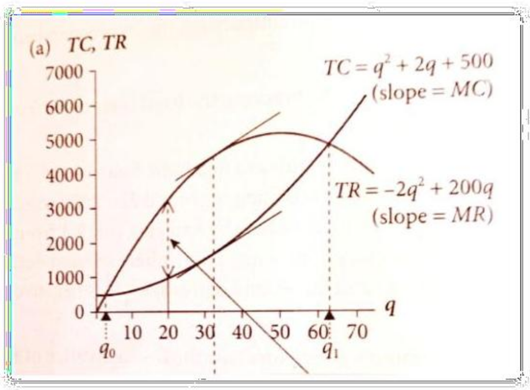
Next we define the firm's profits, which economists usually denote by the Greek letter II (pronounced 'pie' as in apple pie; see appendix 2.1 for the full Greek alphabet). In a commonsense way, we define profits, II, as simply the firm's total revenue from sales minus its total production costs. Thus we have, as a definition:

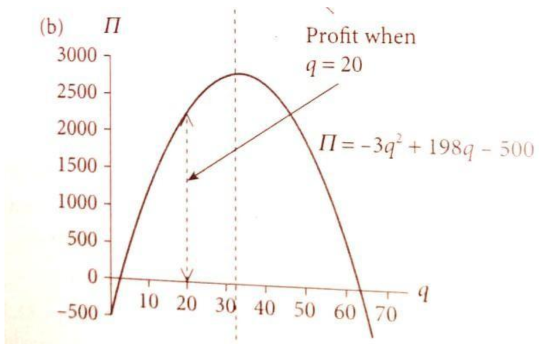
$$II \equiv TR - TC$$

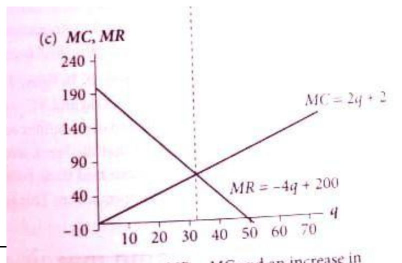
What can we infer about profits from an examination of figure 8.14a? First, at any given level of production and sales, such as q = 20, we can follow the dotted lines to read off the firm's total revenue, which appears from figure 8.14(a) to be about 3100, and its total costs, which appear to be about 900. The difference, TR- TC, between these two values gives us profit at that output, which appears to be about 2200. (All costs, revenues, and profits are, of course, measured in the relevant currency, such as euros.) Thus, in general, at any output the vertical distance between the TR and TC functions in figure 8.14(a) measures II. (You may have noticed that we have implicitly assumed that the quantity produced and the quantity sold are identical. In other words, the firm does not hold any stock of output as a cushion between production and sales.)

Second, from figure 8.14(a) we can also see that there are two levels of output, q_0 and q_1, at which the total revenue and total cost functions intersect, and therefore TR = TC. At these two points, profits are zero. These are known as **break- even points**. While there are two break-even points in this example, other examples can produce only one such point (or, more rarely, three or more). In economic terms, the left-hand break-even point arises from the existence of fixed costs, which give the TC curve a positive intercept. The right-hand break-even point is there because when quantity sold is large, the firm's costs are

is large, the firm's costs are







When $q < 33$, $MR > MC$ and an increase in output increases profits. When $q > 33$, $MR < MC$ and an increase in output decreases profits. Break-even points are q_0 and q_1.

Figure 8.14 Profit maximization under monopoly.

Increasing rapidly due to pressure on plant capacity, while total revenue is declining because the firm can only sell large quantities at relatively low prices due to the constraint of the downward-sloping demand curve.

At any level of output (= quantity sold) between q_0 and q_1, we see from figure 8.14(a) that the TR curve lies above the TC curve, and therefore profits are positive. Conversely, when output is less than q_0 or greater than q_1,TR lies below TC and profits are negative (that is, the firm loses money).

We are now ready to address the key question for the firm: what level of output will yield maximum profits? We can answer this algebraically as follows. We have a total cost function $TC=q^2+2q+500$, and a total revenue function TR=- 2q2+200q. If we substitute these into the profit function defined above, we get

$$\pi = TR - TC = (-2q^2+200q)-(q^2+2q+500)$$

On removing the brackets and simplifying, this expression becomes

$$\pi = -3q^2 +198q-500$$

(Note how, on removing the brackets, the fixed costs of 500 euros are subtracted from the profits, as they should be.)

Hopefully you will recognize this as a quadratic function, which is graphed in figure 8.14(b) Figure 8.14(b) shows profits peaking at around π =2800 when a is about 32 or 33. We can find the exact value of a algebraically by applying rule 7.1 from chapter 7. This rule tells us that Π is at a (local) maximum at any point where the first derivative $\frac{d\Pi}{dq}$ equals zero (the first order condition) and the second derivative $\frac{d^2\Pi}{dq2}$ is negative (the second order condition).

To find the maximum we therefore take the first derivative of Π = $-3q^2+198q-500$, set it equal to zero and solve the resulting equation, giving

$$\frac{d\Pi}{dq} = -6q+198 = 0 \text{ with solution } q = 33.$$

Then we take the second derivative, which is $\frac{d^2\Pi}{dq^2} = -6$. This is negative for all q, including q=30. So at q = 33 we have $\frac{d\Pi}{dq} = 0$ and $\frac{d^2\Pi}{dq^2} < 0$. Thus both the first order and the second order conditions for a maximum of Π are satisfied at q=33, confirming that figure 8.14 is correctly drawn. The maximised level of profits is

$$\Pi = -3q^2+198q-500 = -3(33)^2+198(33)-500=2767$$

In terms of figure 8.14(a), the profit-maximizing output, q = 33, is the point at which the vertical distance between the TR and TC curves reaches its maximum. The significance of the tangents drawn to the TR and TC curves at this point will become clear shortly.

We can also find by algebra the two break-even points. In figure 8.14(a) we can see they occur where TR = TC. Since we have TR $=-2q^2+200q$ and TC$=q^2+29+500$, we can find the break-even points by setting these two equal to one another and solving the resulting equation. Alternatively, we see in figure 8.14(b) that the break-even points occur where Π=0. Since we have Π=-3q2+198q-500 we can find these points by setting this expression equal to zero and solving the resulting equation. This is left to you as an exercise.

Profit maximization using marginal cost and marginal revenue

There is an alternative approach to finding the firm's most profitable output which uses the concepts of marginal cost (MC) and marginal revenue (MR) that we developed in sections 8.4 and 8.10. If you have studied economics before, you may have already met this approach, but don't worry if you haven't as all will be explained.

We can demonstrate this approach by going back to the definition of profits (Π) as simply total revenue (TR) minus total cost (TC):

$$\Pi = TR - TC$$

To find the output at which profits are at a maximum, we look for a point or points where the first derivative $\frac{d\Pi}{dq}$ equals zero and the second derivative $\frac{d^2\Pi}{dq^2}$ is negative.

We begin by finding $\frac{d\Pi}{dq}$ and setting it equal to zero. Given Π = TR-TC, we get

$$\frac{d\Pi}{dq} = \frac{dTR}{dq} - \frac{dTC}{dq} = 0$$

(If you are not completely happy about the validity of this step, refer back to rule D4 of differ-entiation in chapter 6.)

Looking at equation (8.5), on the right-hand side we have $\frac{dTR}{dq}$, which is, by definition, marginal revenue *(MR)*, as defined in section 8.10 above. Similarly we have $\frac{dTC}{dq}$,which is marginal cost *(MC)*, as defined in section 8.4 above. Substituting these into (8.5), the condition for profit maximization becomes

$$\frac{d\pi}{dq} = MR - MC = 0$$

This equation satisfied when *MR = MC.* Thus from equation (8.5) we have learned that the condition $\frac{d\pi}{dq} = 0$ and the condition *MR = MC* are equivalent to one another. In words, the first order condition for maximum profit is that marginal revenue must equal marginal cost. This is true not only for the monopoly firm but also for the perfectly competitive firm, as we will show in the next section.

Returning to the previous example, in figure 8.14(a) the slope of any tangent to the *TR* curve measures *MR*, and the slope of any tangent to the *TC* curve measures *MC*. When *q = 33* , we see that these tangents are parallel, so they have the same slope. Therefore *MR = MC* when *q = 33*.

To the left of *q = 33* at any level of *q* the tangent to the *TR* curve is steeper than the tangent to the *TC* curve, so we must have *MR > MC*. To the right of *q = 33* the opposite is true: *MC > MR.*

Second order conditions

It is tempting to assume that the condition, marginal revenue equals marginal cost, guarantees that profits are maximized. However, the condition *MR = MC* or its equivalent $\frac{d\pi}{dq} = 0$ is only the first order condition for maximum profit. To be sure that the point in question is indeed a point of maximum profit (rather than a minimum or a point of inflection in the profit function) we require the second order condition, $\frac{d^2\pi}{dq^2} < 0$ to be satisfied. By differentiating equation (8.5a) we obtain $\frac{d^2\pi}{dq^2}$ as

$$\frac{d^2\pi}{dq^2} = \frac{dMR}{dq} - \frac{dMC}{dq}$$

and we require this to be negative for a maximum of the profit function. We will postpone consideration of what this implies for *MC* and *MR* until section 8.20 below.

Diagrammatic treatment of the *MC* = *MR* condition

We have just shown how equating marginal cost and marginal revenue gives us an algebraic technique for finding the most profitable output. We now show how the same method can be used diagrammatically.

Returning to the previous example, the *TC* function is $TC = q^2 + 2q + 500$. By differentiating this, we get the *MC* function as

$$MC = \frac{dTC}{dq} = 2q + 2$$

In the same way, the *TR* function is $TR = -2q^2 + 200q$. Differentiating this, we get

$$MR = \frac{dTR}{dq} = -4q + 200$$

The graphs of the *MC* and *MR* functions are shown in figure 8.14c. The $MC = 2q+2$ function is positively sloped, reflecting rising marginal cost in this example. The $MR = -4q + 200$ function is negatively sloped, with a slope of *-4* and an intercept of *200*. It should be no surprise to see that the *MC* and *MR* functions intersect at $q=33$, the point of maximum profit. (Check the algebra for yourself.)

The relationship between price and marginal cost

At the monopolist's profit-maximizing output of $q=33$ we can find the level of *MC* (and therefore of *MR,* since the two are equal) by substituting for *q* in the *MC* function. We get

$MC=2q+2$ which equals *68* when $q=33$

We can also find the price by substituting $q=33$ into the inverse demand function. We get

$p=-2q+200$ which equals *134* when $q=33$

Thus this monopolistic firm is able to sell its output at a price *(134)* that is almost double its marginal production cost *(68)*. This contrasts with perfect competition, where price and marginal cost are equal when profits are maximized, as we shall see in the next section. Economic theory shows this difference between monopoly and competition to be of great significance.

Profit maximization with perfect competition

We will now look at the profit-maximizing behaviour of a perfectly competitive firm and compare the equilibrium with that of the monopolist in the previous example.

EXAMPLE 8.12

We assume that the perfectly competitive firm has the same short-run total cost function as the monopolist in the previous example:

$TC=q^2+2q+500$

This assumption may well be an over-simplification, but it will serve to introduce this area of economics. The perfectly competitive firm can sell all it wishes at the ruling market price, which we will arbitrarily assume to be hat $\acute{P}=72$. The firm's total revenue function is therefore

$$TR=\acute{P}q=72$$

This *TR* function and the *TC* function above are graphed in figure 8.15(a). Comparing this with figure 8.14(a), we see that the difference is that the perfectly competitive firm's *TR* function is linear because it doesn't need to reduce its price in order to sell a large quantity. As in the monopoly case there are two break-even points, now at about $q=8$ and $q=60$.

Since profit is given by the vertical distance between the *TR* and *TC* functions, the profit maximizing output is where this vertical distance is at its greatest, which is at $q=35$ as we will now show.

From the *TC* and *TR* functions we can get the profit function as

$\pi=TR-TC=72q-(q^2+2q+500)$ which simplifies to

$$\pi=-q^2+70q-500$$

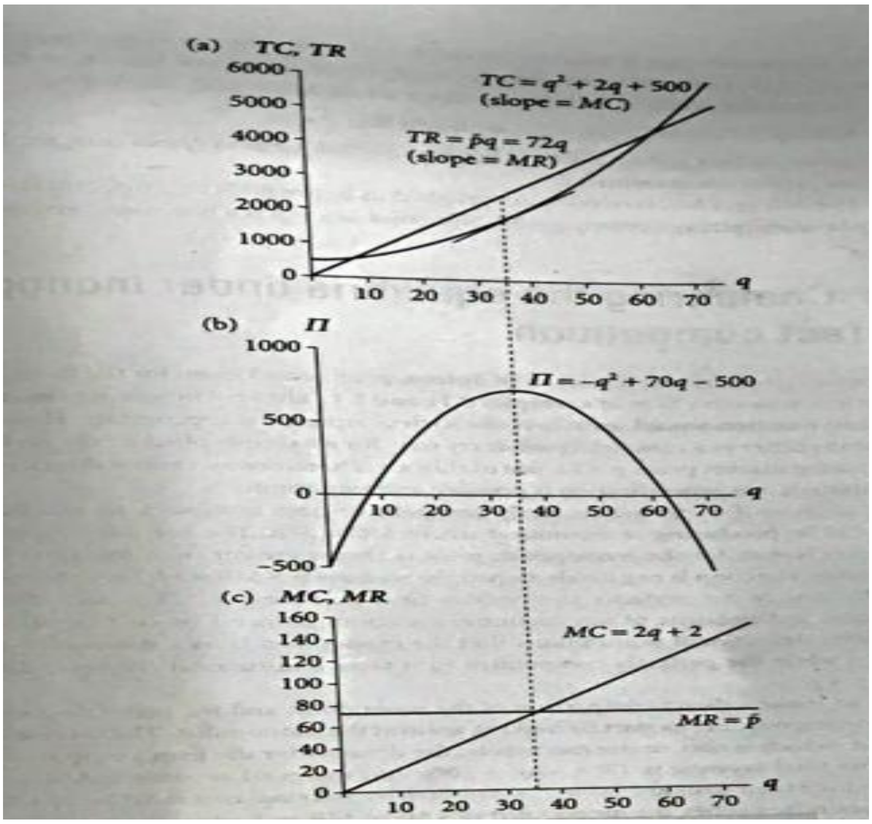
This is graphed in figure 8.15(b). This profit function has the same general shape as in the monopoly case (*example 8.11*), but the parameters are different.

To find maximum profit we set the derivative of the profit function equal to zero. Thus

$\frac{d\pi}{dq}=-2q+70=0$ with solution $q=35$

This is a maximum because $\frac{d^2\pi}{dq^2}$=-2, which is negative when *q=35.* Maximized profits are found from the profit function *q=35.* This gives π=725.

We can alternatively find the profit-maximizing output by setting marginal cost and marginal revenue equal to one another. Looking first for marginal cost, as we have assumed that the perfectly competitive firm and the monopoly firm have the same *TC* function, they obviously have the same *MC* function, found by differentiating the total cost function. Thus, given



When $q<35$, $MR=\acute{P}>MC$ and an increase in output increases. When $q>35$, $MR=\acute{P}<MC$ and an increase in output decreases profits. Break- even points are approx. $q=8$ and $q=62$

Figure 8.15 Profit maximization under perfect competition.

$TC=q^2+2q+500$ we have

$MC=\frac{dTC}{dq}=2q+2$

Looking now for marginal revenue, the perfectly competitive firm's *TR* function is

TR=pq=72q so marginal revenue is

$MR=\frac{dTR}{dq}=p=72$

The key point to see in this last equation is that, for the perfectly competitive firm, marginal revenue and price are necessarily equal, as we saw in section 8.12.

To find the profit-maximizing output we set *MR=MC* which gives

MR=p=72=MC=2q+2 with solution *q=35*

Thus we see that *p=MC*. This is because *p=MR* under perfect competition, and *MR=MC* when profits are maximized.

The *MR* and *MC* functions are graphed in figure 8.15(c). Their intersection gives the profit-maximizing output, *q=35* Note that $MR=\acute{p}$ is a horizontal straight line.

Comparing the equilibrium under monopoly and perfect competition

It is tempting to compare the levels of output, profits, and so on for the monopoly firm and the perfectly competitive firm. The assumption that they have the same cost function would seem to make such comparisons appropriate. However, the demand conditions differ in a completely arbitrary way, for we simply plucked the perfectly competitive firm's ruling market price, $\acute{p}=72$, out of thin air. Therefore we cannot directly compare the two cases, though one generalization is possible and important.

The monopolist and the perfectly competitive firm both seek to maximize their profits and do so by producing at the output where $MC=MR$. The key difference between the two producers is that, for the monopolist, price is always greater than marginal revenue, because the demand function is negatively sloped. So we have $p>MR=MC$ in the case of monopoly.

In the case of the perfectly competitive firm we have $p=MR=MC$. This is an inherent difference, independent of any assumptions about demand or cost conditions, other than the strictly definitional assumptions that the monopolist faces a downward-sloping demand function while the perfectly competitive firm faces a horizontal (inverse) demand function: that is, $p=q$.

We can make a direct comparison of the monopolist and the perfectly competitive firm in the following way. Let us start by looking again at the monopolist. The inverse market demand function (which is also, under monopoly, the demand for the firm's output) is $p=-2q+200$. Therefore total revenue is $TR=-2q^2+200q$ and marginal revenue is $MR=-4q+200$. The monopolist's total costs are $TC=q^2+2q+500$ so marginal cost is $MC=2q+2$.

To maximize profits, the monopolist sets $MC=MR$ and solves the resulting equation for q. This gives

$2q+2=-4q+200$ from which $q=33$

We can then find p by substituting $q=33$ into the inverse demand function, giving

$$p=-2(33)+200=134$$

We will denote these monopolist's equilibrium values as $q_M=33$ and $p_M=134$.

Now let us suppose that the monopolist producer is replaced by a large number of firms, all too small to be able to influence price, and with the same total cost and therefore marginal cost functions as the monopolist. The demand function $p=-2q+200$ remains the market demand *function*, but the demand function for each firm's product is now $p=q$, where p is the ruling market price. So $TR=pq$ and

$MR=\frac{dTR}{dq}=p$.

Each perfectly competitive firm seeks to by setting MC=MR and solving the resulting equation for q. From above, we have MC=2q+2, so MC=MR implies 2q+2=p

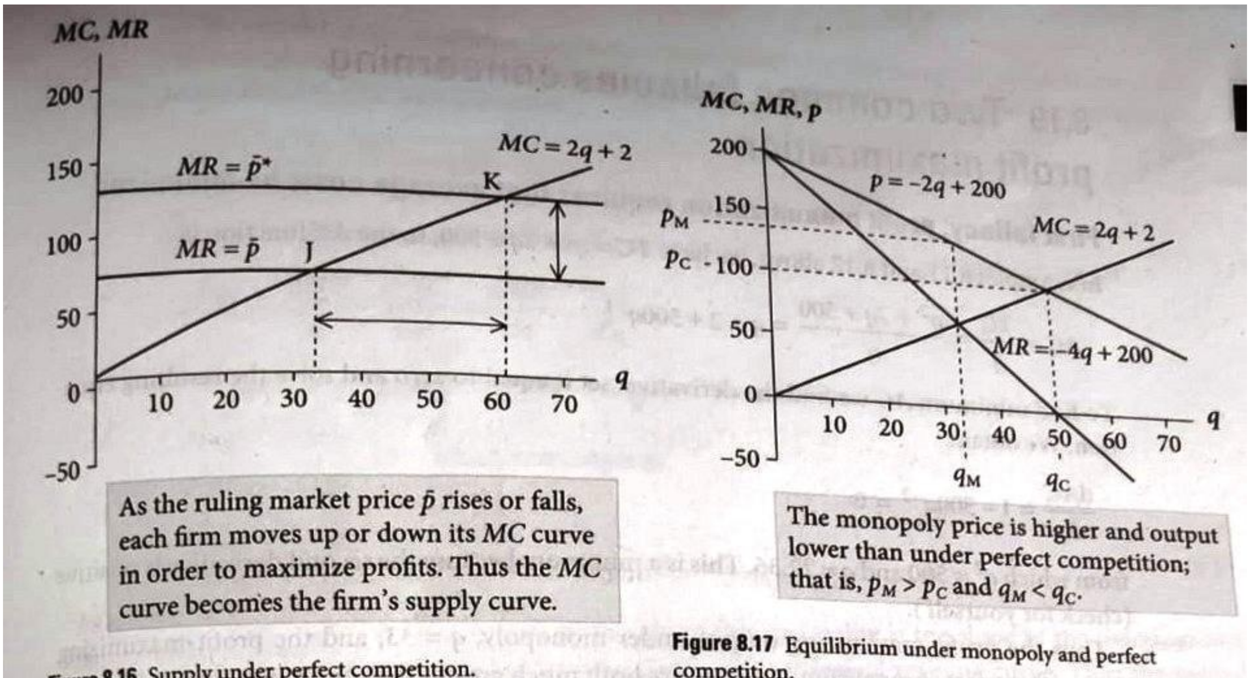


Figure 8.16 Supply under perfect competition.

Figure 8.17 Equilibrium under monopoly and perfect competition.

but this equation has two unknowns, q and p, and therefore has no unique solution. In section 8.17 above we steered around this problem by arbitrarily assuming p = 72.

. To see this, look at figure 8.16. We suppose that a value for p has somehow been determined, and a typical firm is in equilibrium at J with MC = MR. Now suppose p rises to a new level, p*. To restore equilibrium (that is, MC = MR) the firm must move up its marginal cost curve from J to K. Thus the slope of the MC curve tells us the firm's response, in terms of increase in quantity supplied, to a rise in the market price.

But this is exactly what a supply curve tells us; so the firm's MC curve is its supply curve, defined as a relationship between market price and quantity supplied.

Moreover, since this must hold for each and every firm, it must hold for all firms in aggregate, so the aggregate of all firms' MC functions is the industry supply function under perfect competition.

If we assume for simplicity that all firms have the same MC function, we can find the industry supply at any price by simply looking at the quantity supplied by a single firm, at that price, and multiplying this quantity by the number of firms in the industry. In addition, we also know that the equilibrium market price, p, and aggregate quantity sold, q, must satisfy the inverse market demand function $\bar{p}$ = -2q

+200, since otherwise buyers are not in equilibrium. Therefore we must have

$\bar{p}$ = -29+200.

The equilibria under monopoly and perfect competition are shown in figure 8.17. Thus we have a robust conclusion that, under nonopoly, output is lower than under perfect competition (q_M = 33; q_C = 49.50), while price is higher (P_M = 134; P_C = 101). However, to derive this result we assumed that the total cost function and the market or industry demand func-tion are the same under perfect competition and monopoly. Clearly, we have only scratched the surface of a large subject area of economics.

Maximizing excise tax revenue in monopolistic competitive market

In a monopolistically competitive market, the rule for maximizing profit is to set MR = MC, and price is higher than marginal revenue, not equal to it because the demand curve is downward sloping.

Profit is maximized where marginal revenue is equal to marginal cost. In this case, for a competitive firm, marginal revenue is equal to price. So profit is maximized where price is equal to marginal cost or at this point right here.

Minimization of cost

Cost Minimisation is a financial strategy that aims to achieve the most cost-effective way of delivering goods and services to the required level of quality. It is

Cost Minimisation for a Given Output

In the theory of production, the profit-maximising firm is in equilibrium when, given the cost-price function, it maximises its profits on the basis of the least cost combination of factors. For this, it will choose that combination which minimises its cost of production for a given output. This will be the optimal combination for it.

Assumptions:

This analysis is based on the following assumptions:

1. There are two factors, labour and capital.

2. All units of labour and capital are homogeneous.

3. The prices of units of labour (w) and that of capital (r) are given and constant.

4. The cost outlay is given.

5. The firm produces a single product.

6. The price of the product is given and constant.

7. The firm aims at profit maximisation.

8. There is perfect competition in the factor market

Utility Function

This represents the relationship between the quantity of goods consumed and the level of satisfaction or utility derived from them. It can be expressed mathematically as:

U = f(x_1, x_2, \dots, x_n)

2. Constraints:

These are the limitations that individuals face, such as budget constraints. For example, if a person has a limited income, they must…

3. Optimization:

Utility maximization involves choosing the combination of goods that provides the highest utility. This can be done using calculus (Lagrange multipliers) or graphical methods (indifference curves and budget lines).

4. Indifference Curves:

These curves represent combinations of goods that provide the same level of utility to the consumer. The consumer's goal is to reach the highest possible indifference curve given their budget constraint

5. Budget Line:

This line represents all the combinations of goods that a consumer can afford given their income and the prices of goods. Important to remember that cost minimization is not about reducing quality or short- changing customers – it always remains important to meet customer needs.

Cost Minimisation for a Given Output

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This analysis is based on the following assumptions:

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2. All units of labour and capital are homogeneous.
3. The prices of units of labour (w)and that of capital (r) are given and constant.

4. The cost outlay is given.

5. The firm produces a single product.

6. The price of the product is given and constant.

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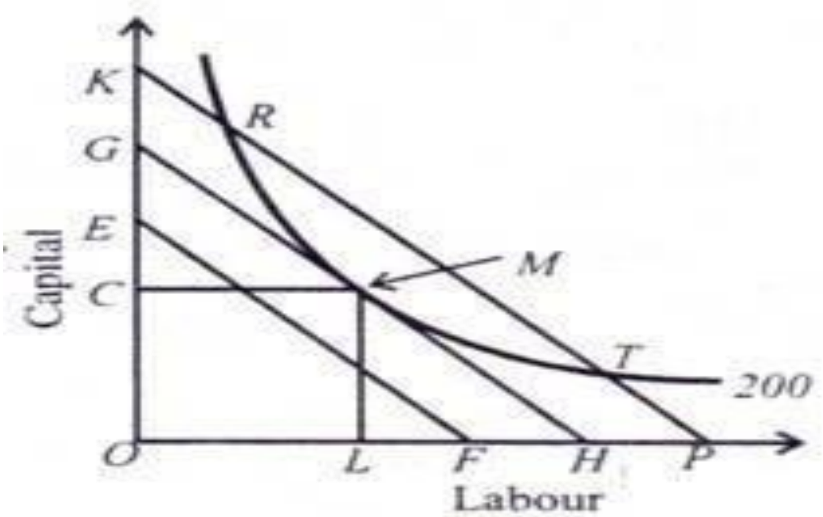


Fig. 15

Explanation:

Given these assumptions, the point of least-cost combination of factors for a given level of output is where the is quant curve is tangent to an is cost line. In Figure 15, the is cost line GH is tangent to the is quant 200 at point M. The firm employs the combination of OC of capital and OL of labour to produce 200 units of output at point M with the given cost-outlay GH.

At this point, the firm is minimising its cost for producing 200 units. Any other combination on the is quant 200, such as R or T, is on the higher is cost line KP which shows higher cost of production. The is cost line EF shows lower cost but output 200 cannot be attained with it. Therefore, the firm will choose the minimum cost point M which is the least-cost factor combination for producing 200 units of output. M is thus the optimal combination for the firm.

The point of tangency between the is cost line and the is quant is an important first order condition but not a necessary condition for the producer's equilibrium. TherearetwoessentialKorsecondorderconditionsfortheequilibriumofthe firm:

1. The first condition is that the slope of the is cost line must equal the slope of the is quant curve. The slope of the is cost line is equal to the ratio of the price of labour (w) and the price of capital(r). The slope of the is quant curve is equal to the marginal rate of technical substitution of labour and capital (MRTSLK) which is, in turn, equal to the ratio of the marginal product of labour to the marginal product of capital (MPL/MPK' condition for optimality can be written as. w/rMPL/MPK= MRTSLK
2. The second condition is that at the point of tangency, the is quant curve must be convex to the origin. In other words, the marginal rate of technical substitution of labour for capital (MRTSLK) must be diminishing at the point of an agency for equilibrium to be Stable. In Figure 16, S cannot be the point of equilibrium for the is quant IQ1 is concave where it is tangent to the is cost line GH. At point S, the marginal rate of technical substitution between the two factors increases if move to the right or left on the curveIQ1.

Stable. In Figure 16, S cannot be the point of equilibrium for the is quant IQ1 is concave where it is tangent to the is cost line GH. At point S, the marginal rate of technical substitution between the two factors increases if move to the right or left on the curveIQ1.

Moreover, the same output level can be produced at a lower cost AB or EF and there will be a comer solution either at C or F. If it decides to produce at EF cost, it can produce the entire output with only OF labour. If, on the other hand, it decides to produce at a still lower cost CD, the entire output can be produced with only OC capital.

Both the situations are impossibilities because nothing can be produced either with only labour or only capital. Therefore, the firm can produce the same level of output at point M, where the is quant curve IQ is convex to the origin and is tangent to the is cost line GH. The analysis assumes that both the is quant's represent equal level of output, IQ = IQ1.

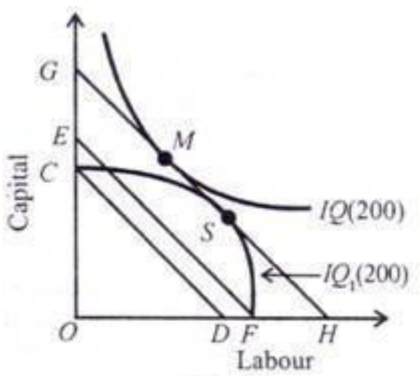


Fig. 16

Output-Maximisation for a Given Cost:

The I so maximises its profits by maximising its output, given its cost outlay and the prices of the two factors. This analysis is based on the same assumptions, as given above. The conditions for the equilibrium of the firm are the same, as discussed above.

The firm is in equilibrium at point P where the is quant curve 200 is tangent to the is cost line CL in Figure 17. At this point, the firm is maximising its output level of 200 units by employing the optimal combination of OM of capital and ON of labour, given its cost outlay CL.

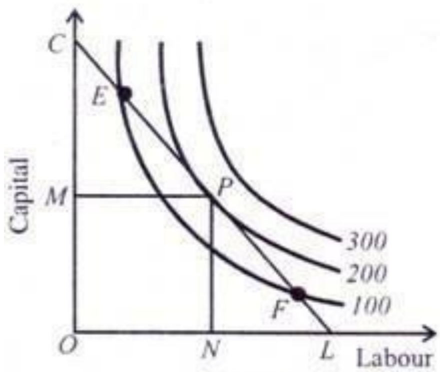


Fig. 17

But it cannot be at points E or F on the is cost line CL, since both points give a smaller quantity of output, being on the is quant 100, than on the is quant 200. The firm can reach the optimal factor com-bination level of maximum output by moving along the is cost line CL from either point E or F to point P.

This movement involves no extra cost because the firm re-mains on the same is cost line. The firm cannot attain a higher level of output such as is quant 300 because of the cost constraint. Thus the equilibrium point has to be P with optimal factor combination OM+ ON. At point P, the slope of the is quant curve 200 is equal to the slope of the is cost line CL. It implies w/r = MPL/MPK = MRTSLK.

1. The second condition is that the is quant curve must be convex to the origin at the point of tangency with the is cost line, as explained above in terms of Figure 16.

Cost Minimization

Cost minimization is a strategy or approach used by businesses and organizations to reduce expenses and optimize resources while maintaining productivity and quality. It involves identifying areas where costs can be cut or minimized without compromising the efficiency or effectiveness of operations. This can include measures such as:

1. Streamlining Processes:

Identifying and eliminating unnecessary steps or inefficiencies in workflows and operations.

2. Negotiating with Suppliers:

Negotiating better deals with suppliers for raw materials, equipment, or services to lower procurement costs.

3. Optimizing Inventory:

Managing inventory levels efficiently to reduce holding costs while ensuring adequate stock to meet demand.

4. Utilizing Technology:

Implementing technology solutions such as automation, cloud-based tools, and software systems to improve efficiency and reduce manual labor costs.

5. Energy Efficiency:

Implementing energy-saving measures to reduce utility costs, such as using energy-efficient equipment and optimizing lighting and heating/cooling systems.

6. Outsourcing and Off shoring:

Exploring outsourcing or off shoring options for non-core functions to lower labor costs while maintaining quality standards.

7. Training and Development:

Investing in training programs to enhance employee skills and productivity, reducing errors and rework costs.

8. Monitoring and Analysis:

Regularly monitoring expenses, analyzing cost drivers, and identifying opportunities for further cost reduction.

9. Bench marking:

Comparing costs and performance metrics with industry peers or best practices to identify areas for improvement.

cost minimization is a continuous process that requires on going evaluation, adaptation and decision- making to achieve sustainable cost reductions while supporting organizational goals and objectives.

UNIT-III

OPTIMIZATION TECHNIQUES WITH CONSTRAINTS

Function of several variables

Definition

A function of several variables is a relation between a set of input variables and a set of output variables. It is a mapping from a subset of $\mathbb{R}^n$ to $\mathbb{R}^m$, where n is the number of input variables and m is the number of output variables.

Basic Properties

1. Domain: The set of all possible input values.
2. Range: The set of all possible output values.
3. Continuity: A function is continuous if it can be drawn without lifting the pencil from the paper.
4. Differentiability: A function is differentiable if it has a tangent plane at every point in its domain.

Types of Functions

1. Linear Functions: Functions that can be written in the form $f(x, y) = ax + by + c$.
2. Quadratic Functions: Functions that can be written in the form $f(x, y) = ax^2 + bxy + cy^2 + dx + ey + f$.
3. Polynomial Functions: Functions that can be written in the form $f(x, y) = \sum\sum a_{ij} x^i y^j$.

Applications

1. Economics: Functions of several variables are used to model economic systems, including production functions, utility functions, and cost functions.
2. Physics: Functions of several variables are used to model physical systems, including gravitational potential, electric potential, and magnetic fields.
3. Computer Science: Functions of several variables are used in machine learning algorithms, including neural networks and decision trees.
4. Engineering: Functions of several variables are used to design and optimize systems, including control systems, signal processing systems, and communication systems.

Examples of Functions of Several Variables

1. Production Function: Q = f(K, L), where Q is output, K is capital, and L is labor.
2. Utility Function: U = f(x, y), where U is utility, x is consumption of good x, and y is consumption of good y.
3. Cost Function: C = f(Q, w, r), where C is cost, Q is output, w is wage rate, and r is rental rate.
4. Gravitational Potential: V = f(x, y, z), where V is gravitational potential, x, y, and z are coordinates in space.

Operations on Functions of Several Variables

1. Addition: f(x, y) + g(x, y)

2. Subtraction: f(x, y) - g(x, y)

3. Multiplication: $f(x, y) \times g(x, y)$

4. Division: $f(x, y) \div g(x, y)$

Partial Derivatives

1. Definition: The partial derivative of a function $f(x, y)$ with respect to x is denoted by $\partial f/\partial x$ and is defined as the derivative of f with respect to x, treating y as a constant.
2. Notation: $\partial f/\partial x$, $\partial f/\partial y$
3. Geometric Interpretation: The partial derivative of a function at a point represents the rate of change of the function in the direction of the variable with respect to which the derivative is taken.

Total Derivatives

1. Definition: The total derivative of a function $f(x, y)$ is denoted by df and is defined as the sum of the partial derivatives of f with respect to x and y.
2. Notation: $df = \partial f/\partial x \, dx + \partial f/\partial y \, dy$
3. Geometric Interpretation: The total derivative of a function at a point represents the rate of change of the function in an arbitrary direction.

Partial and Total Derivatives

Introduction

So far we concerned ourselves with functions of one independent variable: for example, while introducing the technique of differentiation, we thought of total utility y as a function of consumption of one commodity x, so that $y = f(x)$. Also, all our rules of differentiation assumed y as function of only one variable x.

But the total utility or any quantity may in fact be a function of two or more independent variables. We can cite many such examples in economics. If z tonnes of wheat is produced on y acres of land with x number of labourers; z is the function of both x and y, that is: z = f(x,y). If consumer purchases meat x and bread y for his lunch, then his total utility u will depend on amounts of both x and y consumed by him so that: u = f(x,y). Similarly, we can think of total utility U being function of amounts of different commodities x_1, x_2, x_3 ...., x_n consumed by the consumer in a form of function :

$$U = f(x_1, x_2, x_3, \ldots \ldots \ldots \ldots \ldots \ldots x_n)$$

where x_1, x_2, $x_3 \ldots . x_n$ are all independent of one another and that each can vary by itself without affecting others.

We consider a simple case of 'U' being dependent on two variables:

$$U = f(x,y)$$

If the variable x undergoes change (assuming that consumption of only one commodity undergoes change) while y remains constant (as it was assumed previously), there will be a corresponding change in U, say it is ΔU, then :

U+ ΔU = f(x + Δx, y) AU_f(x + Ax, y) – f(x,

or $\Delta U = f(x + \Delta x, y) - U$

or $\Delta U = f(x + \Delta x, y) - f(x, y)$

or $\frac{\Delta U}{\Delta x} = \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$

Now we take limit of $\frac{\Delta U}{\Delta x}$ as Δx tends to zero to find the derivatives:

$$\operatorname*{Ltd}_{\Delta x \to 0} \frac{\Delta U}{\Delta x} = \operatorname*{Ltd}_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$$

we call this specific derivative as *'Partial Derviative'* of U with respect to x. We call it "Partial" derivative to indicate that y (or all other variables in case tire are more than two variables) in the function has been held constant.

Such partial derivative is assigned different symbol to indicate that this is the partial derivative with respect to x and that other variable y has been regarded as fixed.

In place of letter 'd', we use symbol ∂ and write partial derivative of U with respect to x as:

$\frac{\partial U}{\partial x}$ (not as $\frac{dU}{dx}$)

Thus,

$$\operatorname*{Ltd}_{\Delta x \to 0} \frac{\Delta U}{\Delta x} = \frac{\partial U}{\partial x} \operatorname*{Ltd}_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$$

Similarly, we can have the partial derivative of U with respect to y as:

$$\operatorname*{Ltd}_{\Delta y \to 0} \frac{\Delta U}{\Delta y} = \frac{\partial U}{\partial y} \operatorname*{Ltd}_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y}$$

Partial derivatives are also depicted by symbols as f_x, f_y, f_z,.................. where subscript indicates which independent variable (alone) is being allowed to vary.

For example,

$\frac{\partial U}{\partial x}$ can also be represented by f_x and $\frac{\partial U}{\partial x}$ by f_y (Remember that in case of $\frac{dy}{dx}$ we used the symbol f ' (x)].

Technique of Partial Differentiation

The process of taking partial derivative is called partial differentiation and it differs from previously discussed differentiation primarily in that we hold and treat all the independent variables constant except the one which is assumed to vary.

The practical technique of partial differentiation is illustrated by the following examples:

Ex. 1. Given function is U = 5x - 6y + 8 and we are required to find partial derivatives.

There can be only two partial derivatives.

(1) U with respect to x when y is held constant.

that is, $\frac{\partial U}{\partial x}$ or,

(2) U with respect to y wisely is held constant

that is, $\frac{\partial U}{\partial y}$ or, f_y

In case of (1), since y is held constant it is treated as a constant term during

differentiatica;

while in case of (2), since x is held constant it is treated as a constant term during differentiation; thus we have

(1) $\frac{\partial U}{\partial x} = f_x = 5 - 0 + 0 = 5$ (6y and +8 are treated as constant terms).

(2) $\frac{\partial U}{\partial y} = f_y = 0 - 6 + 0 = -6$ (5x and +8 are treated as constant terms).

Similarly if $Z = ax + by + c$ then

$$\frac{\partial Z}{\partial x} = a, \qquad \frac{\partial Z}{\partial y} = b.$$

Ex. 2. Find the partial derivatives of $Z = 4x^2 + 4xy + y^2$

We have; $\frac{\partial Z}{\partial x} = 8x + 4y$ and,

$$\frac{\partial Z}{\partial y} = 4x + 2y$$

Ex. 3. Given: $Z=(x+4)(2x+5y)$; find partial derivatives.

Partial derivatives can be obtained by the use of product rule.

Thus we have, $fx = (x+4)(2) + (1)(2x + 5y)$

$$fx = 2x+8+2x + 5y = 4x + 5y + 8$$

Similarly, by holding x as constant

$$f = (x + 4)(5) + (0)(2x + 5y) = 5x + 20$$

Ex. 4. $Z = x^3 e^{2y}$; find partial derivatives

$$f_x = 3x^2 e^{2y}$$

$$fy = x^3 2e^{2y} = 2x^3 e^{2y}$$

Ex. 5. $Z = \frac{x+y}{2x+5y}$; find partial derivatives.

$$\frac{\partial Z}{\partial x} = f_x = \frac{(1)\,(2x+5)-\,(2)\,(x+y)}{(2x+5y)^2} = \frac{3y}{(2x+5y)^2}$$

$$\frac{\partial Z}{\partial y} = f_y = \frac{(1)\,(2x+5)-(5)\,(x+y)}{(2x+5y)^2} = \frac{-3y}{(2x+5y)^2}$$

Ex. 6. $Z = \frac{5x^2}{5x-y+4}$; find partial derivatives.

$$f_x = \frac{10x(5x-y+4)-(-1)5\,(5x^2)}{(5x-y+4)^2} = \frac{25x^2-10xy+40x}{(5x-y+4)^2}$$

$$f_y = \frac{(0)(5x-y+4)-5\,(5x^2)}{(5x-5+4)^2} = \frac{5x^2}{(5x-y+4)^2}$$

Partial Derivatives of Second Order

We can also find the partial derivatives of second or higher orders. The process of partial derivation can be repeated until the partial derivative happens to be the function of any of the independent variables in the original function.

For example:

(1) If $\frac{\partial Z}{\partial x}$ (the partial derivative of Z with respect to x) happens to be the function of x and y, it could be differentiated partially with respect to x and written as

$$\frac{\partial Z}{\partial x}\left(\frac{\partial Z}{\partial x}\right) \text{ and denoted by } \frac{\partial^2 Z}{\partial x^2} \text{ or. } f_{xx}$$

In other words, fxx $=\left(\frac{\partial^2 Z}{\partial x^2}\right)$ is the second partial derivative obtained by partial derivation, first with respect to x and then again with respect to x.

(2) If $\frac{\partial Z}{\partial x}$ is a function of x and y, it can be differentiated partially with respect to y.

This is denoted by $\frac{\partial}{\partial y}\left(\frac{\partial Z}{\partial x}\right)$ or $\frac{\partial^2 Z}{\partial y \partial x}$ or f_{yx} in other words, f_{xy} is the second order partial derivative obtained by partial derivation; first with respect to x and then with respect to y.

(3) Similarly, if $\frac{\partial Z}{\partial y}$ (the partial derivative of Z with respect to y) happens to be the function of both x and y, it could be partially differentiated with respect to x and y, so that when it is partially differentiated with respect to x

we have

$$\frac{\partial}{\partial x}\left(\frac{\partial Z}{\partial y}\right) \text{ denoted by } \frac{\partial^2 Z}{\partial x \partial y} \text{ or, } f_{xy}$$

(4) We can also have

$$\frac{\partial}{\partial y}\left(\frac{\partial Z}{\partial y}\right) \text{ denoted by } \frac{\partial^2 Z}{\partial y^2} \text{ or, } f_{yy}$$

Thus, it may be noted that we obtain two partial derivatives from each of the partial derivatives of Z; or we obtain second order partial derivatives of the original function:

These are: f_{xy} , f_{yx} , f_{xy} and f_{yy}

Cross Partial Derivatives

Second order partial derivatives: f_{xy} and f_{yx} are known as cross partial derivatives. It may be noticed that the cross partial derivatives

f_{xy} $\left(= \frac{\partial^2 Z}{\partial x \partial y}\right)$ and,

f_{yx} $\left(= \frac{\partial^2 Z}{\partial x \partial y}\right)$ are different only in the order in which Z has been differentiated partially.

(1) $\frac{\partial^2 Z}{\partial x \partial y}$ or f_{xy} indicates that the given function Z is first partially differentiated with respect to x and then with respect to y.

(2) $\frac{\partial^2 Z}{\partial y \partial x}$ or f_{xy} indicates that the given function Z is first partially

differentiated with respect to y and then with respect to x.

However, it can be shown that under certain conditions (relating to the continuity of the function) the cross partial derivatives are identical in value, that is:

$\frac{\partial^2 Z}{\partial y \partial x} = \frac{\partial^2 Z}{\partial x \partial y}$; or, $f_{yx} = f_{xy}$

In other words, the order of the partial derivation does not make any difference in the process of partial differentiation.

We take a few examples to show this property of partial differentiation.

Ex. 7. $Z = \frac{x+4}{2x+5y}$

$$\frac{\partial Z}{\partial y} = f_x = \frac{5y-8}{(2x+5y)^2}$$

Since $\frac{5y-8}{(2x+5y)^2}$ is the function of both x and y, we can find second order derivatives as follows:

$$\frac{\partial^2 Z}{\partial x \partial x} = f_{xx} = \frac{0(2x+5y)^2 - 2(2x+5y)(2)(5y-8)}{(2x+5y)^4} = \frac{-4(5y-8)}{(2x+5y)^3}$$

$$\frac{\partial^2 Z}{\partial x \partial x} = f_{yx} = \frac{5(2x+5y)^2 - 2(2x+5y)(5)(5y-8)}{(2x+5y)^4} = \frac{10x-25y+80}{(2x+5y)^3}$$

Also,

$$\frac{\partial Z}{\partial y} = f_y = \frac{-5x-20}{(2x+5y)^2}$$

$$\frac{\partial^2 Z}{\partial x \partial y} = f_{xy} = \frac{-5(2x+5y)^2 - 2(2x+5y)(2)(-5x-20)}{(2x+5y)^4} = \frac{10x-25y+80}{(2x+5y)^3}$$

Hence : $f_{xy} = f_{yx}$ and

$$\frac{\partial^2 Z}{\partial y \partial y} = f_{yy} = \frac{0(2x+5y)^2 - 2(2x+5y)(5)(-5x-20)}{(2x+5y)^4} = \frac{50(x+4)}{(2x+5y)^3}$$

Ex. 8. Given: $Z = x^3 e^{2y}$. Find all partial derivatives of the second order:

$f_x = 3x^2 e^{2y}$

$f_{xx} = 6x (e^{2y}) = 6xe^{2y}$

$f_{yx} = 6x^2 e^{2y} + (0) (e^{2y}) = 6x^2 e^{2y}$

$f_y = 2x^3 e^{2y}$

$f_{xy} = 2x^3 (0) + 6x^2 e^{2y} = 6x^2 e^{2y}$

$f_{yy} = 2x^3 e^{2y} (2) + (0) (e^{2y}) = 4x^3 e^{2y}$

Ex 8.1 Find the first and second order partial derivatives of each of the following functions, also verify them:

$\frac{\partial^2 Z}{\partial x \partial y} = \frac{\partial^2 Z}{\partial y \partial x}$ (in the examples up to 8)

1. $Z = 2x^2+5x^2y + xy^2 + y^2$
2. $Z = 12 - x^2 - y^2 + xy$
3. $Z = (2x+5y) e^y$
4. $Z = 3 \sqrt{xy}$
5. $Z = (x^2+2xy + y^2).e^2$
6. $Z = x + ye^{-x}$
7. $Z = \frac{2x^2}{2x-3y+1}$
8. $Z = \frac{x^3+ y^3}{}$

$x^2 + y^2$

9. If $Z = \log(1 + x^2) + y^2$, find f_{xx}, f_{yy} and f_{xy} in each of the above cases.

Partial Derivatives of Functions of More Than Two Variables

When defining the partial derivatives we considered a simple case of Actions involving only two variables, e.g., Z = f(x, y). But we can have a function involving more than two variables; [U = f(x, y, z)]. Partial derivatives In this case may be defined as:

(1) f_{xy} $\left(= \frac{\partial U}{\partial x}\right)$ represents the change in U with respect to x

when y and z are being held as constants.

(2) f_y $\left(= \frac{\partial U}{\partial y}\right)$ represents the rate of change in U with respect to y

when x and z are being held as constant.

(3) f_z $\left(= \frac{\partial U}{\partial y}\right)$ represents the rate of change in U with respect to z

when x and y are being held as constants.

The second order partial derivatives can be obtained as in the case of functions involving two variables.

Since each of the three partial derivatives of first order will give rise to three second order partial derivatives, we shall have a group of nine second order partial derivatives as follows:

Original function: U = (x, y, z)

First order partial derivatives ***Second order partial***

derivatives

f_x	f_{xx}	f_{yx}	f_{zx}	
f_y	f_{yy}	f_{xy}	f_{zy}	Total 9 partial derivatives
f_z	f_{zz}	f_{xz}	f_{yz}	

Of which f_{xy}, f_{yx}, f_{zy}, f_{xz} and f_{fzx} are cross partial derivatives. Again the following will be identical:

$$f_{xy} = f_{yx},\ f_{yz} = f_{yz} \text{ and } f_{xz} = f_{zx}$$

Therefore, we will have only six distinct values of the nine second order partial derivatives; three of which will be direct partial derivatives, viz. f_{xx} f_{yy} and f_{zz} and three cross partial derivatives; each equal, respectively to f_{yx}, f_{zy} and f_{xz}

Young's Theorem:

Young's Theorem, also known as the Young-Laplace equation, is a fundamental concept in mathematics and physics that describes the behavior of surfaces under stress. The theorem states that the curvature of a surface is proportional to the difference in pressure across the surface. Mathematically, this is expressed as $\Delta P = 2\gamma H$, where ΔP is the pressure difference, γ is the surface tension, and H is the mean curvature of the surface.

The theorem was first proposed by Thomas Young in 1805, and later developed further by Pierre-Simon Laplace. It has since been widely applied in various fields, including physics, engineering, and biology. One of the key implications of Young's Theorem is that it provides a quantitative relationship between the curvature of a surface and the forces acting upon it. This has important consequences for our understanding of phenomena such as capillary action, soap bubbles, and cell membrane dynamics.

In addition to its practical applications, Young's Theorem also has significant theoretical implications. For example, it provides a fundamental link between the geometry of a surface and the physical forces that act upon it. This has led to important advances in our understanding of the behavior of complex systems, such as foams, emulsions, and biological tissues. Furthermore, Young's Theorem has also been used to study the behavior of surfaces at the nanoscale, where the effects of surface tension and curvature can be particularly significant.

Overall, Young's Theorem is a fundamental concept that has far-reaching implications for our understanding of the behavior of surfaces and interfaces. Its applications are diverse, ranging from the study of biological systems to the development of new materials and technologies. As such, it remains an important area of research and study in mathematics, physics, and engineering.

Young's Theorem with equations:

Young's Theorem

Young's Theorem, also known as the Young-Laplace equation, describes the behavior of surfaces under stress. It states that the curvature of a surface is proportional to the difference in pressure across the surface.

Equation

The Young-Laplace equation is given by:

$\Delta P = 2\gamma H$

where:

- ΔP is the pressure difference across the surface

- γ is the surface tension
- H is the mean curvature of the surface

Components of the Equation

1. Pressure Difference (ΔP): The difference in pressure across the surface.
2. Surface Tension (γ): The energy per unit area of the surface.
3. Mean Curvature (H): The average curvature of the surface.

Assumptions

1. Isotropic Surface: The surface has the same properties in all directions.
2. Homogeneous Surface: The surface has the same properties throughout.
3. Small Deformations: The surface deformations are small compared to the size of the surface.

Applications

1. Capillary Action: The rise of a liquid in a narrow tube.
2. Soap Bubbles: The formation and behavior of soap bubbles.
3. Cell Membrane Dynamics: The behavior of cell membranes under stress.

Limitations

1. Non-Linear Effects: The equation assumes linear behavior, but non-linear effects can occur at high pressures or curvatures.
2. Anisotropic Surfaces: The equation assumes isotropic surfaces, but anisotropic surfaces can exhibit different behavior.

Extensions

1. Non-Newtonian Fluids: The equation can be extended to non-Newtonian fluids, which exhibit non-linear behavior.

2. Dynamic Surfaces: The equation can be extended to dynamic surfaces, which exhibit time-dependent behavior.

HOMOGENEOUS FUNCTIONS

A homogeneous functions is a special type of function frequently used in economics.

A function is homogeneous of degree n if when each of its variables is replaced by k^n times the variable, the new function is k^n times the original function.

If $z=f(x, y)$ is homogeneous of degree n,

then $f(kx, ky)=k^n z=k^n f(x, y)$

Ex. 61. $f(x, y) = x^2 - 3xy + 5y^2$ is a homogeneous function of degree 2 because

$$f(kx, ky) = k^2x^2 - 3kx.\, ky + 5k2y2$$
$$= k^2 (x^2 - 3xy + 5y^2) = k^2 f(x, y)$$

Ex. 62. $f(L, K)$ is homogeneous of degree α if

$$f(aL, aK) = a^\alpha f(L, K)$$

Ex. 63. $f(x, y, z) = \frac{y2}{z} - \frac{3x2}{y}$ is homogeneous of degree one.

$$f(kx, ky, kz) = \frac{k^2 y^2}{kx} - \frac{3k^2x^2}{ky} = k\left(\frac{y^2}{z} - \frac{3y^2}{y}\right)$$

$$= kf(x, y, z)$$

A linearly homogeneous function of degree one is such that if

$$z = f(x, y),\ f(kx, ky) = kf(x, y)$$

Ex. 64. $f(x, y) = x + y$ is linearly homogeneous.

(i) $f(lx, ly) = lx + ly = l(x + y) = lf(x, y)$

(ii) $f(x, y) = \sqrt{ax^2 + by^2}$ is linearly homogeneous

$f(kx, ky) = \sqrt{ak^2x^2 + bk^2y^2}$

$= k\sqrt{ax^2 \quad by^2} = kf(x, y)$

Ex. 65. $f(x, y, z) = \frac{3z}{x} - \frac{5x}{y}$

$= \frac{3kz}{kx} - \frac{3kx}{ky}$

$= \frac{3z}{x} - \frac{5x}{y}$

$= k^\circ f(x, y, z)$

Here f(x, y, z) is homogeneous of degree zero.

Ex. 66. Show that the following functions are linearly homogeneous

(i) $z = ax + by$ (ii) $z = \sqrt{ax^2 + bxy + cy^2}$

(iii) $z = (ax^2+bxy + su^2)/(x + y)$

COBB DOUGLAS PRODUCTION FUNCTION

If L. stands for labour, K for capital and α, ß are constants such that α + ß = 1 then in the Cobb Douglas production function, replacing L, K by pL, pK

$Y = aL^{\alpha} K^{ß}$

We have

$a. (pL)^{\alpha}. (pK)^{ß} = a. p^{\alpha} L^{ß} . p^{ß} K^{ß}$

$= a. p^{\alpha+ß} L^{\alpha} K^{ß} = p. aL^{\alpha} L^{ß} = p^{Y}$ since $\alpha + ß = 1$

Hence the Cobb Douglas production function is linearly homogeneous.

Ex. 67. Show that (i) $Y = \left(\frac{\alpha}{L^r} + \frac{\beta}{L^r}\right)^{-1/r}$

(ii) $Y = a\left[\delta K^{-r} + (1-\delta)L^{-r}\right]^{-1/r}$

are homogeneous of degree 1.

Returns to scale

Let Y = f (K, L) be a linearly homogeneous production function in labor L capital K.

Raising all inputs of L, K c-fold will raise the output c-fold.

Since f(cK, cL) = cY

In other words, a linearly homogeneous production function implies constant return to scale in production. If no input is used there is no output. If inputs are doubled, o is doubled.

EULER'S THEOREM

The value of a linearly homogeneous function is the sum of the products of the independent variables and the corresponding partial derivatives. If x = x (u, v), is homogeneous of degree n, Euler's theorem states that

$$u\frac{\partial x}{\partial u} + v\frac{\partial x}{\partial v} = nx\,(u,v)$$

If n = 1, we have $u\frac{\partial x}{\partial u} + v\frac{\partial x}{\partial v} = x\,(u,v) = x$ say

The result can be extended to any number of variables. In economics, Euler's theorem is applied to marginal productivity theory and various other situations

Constrained Optimization

Constrained optimization is the process of finding the maximum or minimum of a function subject to constraints on the variables.

Types of Constraints

1. Equality Constraints: Constraints that must be satisfied exactly, e.g., $x + y = 2$.
2. Inequality Constraints: Constraints that must be satisfied within a certain range, e.g., $x + y \leq 2$.
3. Mixed-Integer Constraints: Constraints that involve both integer and continuous variables.

Methods for Constrained Optimization

1. Lagrange Multipliers: A method that uses a Lagrange multiplier to enforce equality constraints.
2. Karush-Kuhn-Tucker (KKT) Conditions: A set of conditions that are necessary and sufficient for a point to be a local minimum of a constrained optimization problem.
3. Quadratic Programming: A method that solves a quadratic optimization problem subject to linear constraints.
4. Sequential Quadratic Programming (SQP): A method that solves a nonlinear optimization problem subject to nonlinear constraints using a sequence of quadratic programming subproblems.

Steps for Solving a Constrained Optimization Problem

1. Define the Problem: Define the objective function and constraints.
2. Choose a Method: Choose a method for solving the problem, such as Lagrange multipliers or quadratic programming.
3. Find the Optimum: Use the chosen method to find the optimum solution.
4. Check the Solution: Check the solution to ensure that it satisfies the constraints and is a local minimum.

Examples of Constrained Optimization Problems

1. Portfolio Optimization: Find the optimal portfolio of stocks and bonds that maximizes return subject to a constraint on risk.
2. Resource Allocation: Allocate resources to different projects to maximize profit subject to constraints on budget and personnel.
3. Design Optimization: Design a system to minimize cost subject to constraints on performance and safety.

Here are the notes on the Lagrangian Multiplier Technique: Lagrangian Multiplier Technique

Introduction

The Lagrangian Multiplier Technique is a method used to solve constrained optimization problems. It is a powerful technique that can be used to find the maximum or minimum of a function subject to one or more constraints.

Lagrangian Function

The Lagrangian function is defined as:

$L(x, \lambda) = f(x) - \lambda(g(x) - c)$

where:

- $f(x)$ is the objective function
- $g(x)$ is the constraint function
- c is the constraint value
- λ is the Lagrange multiplier
- x is the vector of decision variables

Lagrange Multiplier Technique

The Lagrange Multiplier Technique involves the following steps:

1. Define the Lagrangian function: Define the Lagrangian function using the objective function, constraint function, and Lagrange multiplier.
2. Compute the gradient of the Lagrangian function: Compute the gradient of the Lagrangian function with respect to the decision variables and the Lagrange multiplier.
3. Set the gradient equal to zero: Set the gradient of the Lagrangian function equal to zero and solve for the decision variables and the Lagrange multiplier. Check the second-order conditions: Check the second-order conditions to ensure that the solution is a maximum or minimum.

Example

Suppose we want to maximize the function:

$f(x, y) = 2x + 3y$

subject to the constraint:

$x + y = 4$

Using the Lagrange Multiplier Technique, we can define the Lagrangian function as:

$L(x, y, \lambda) = 2x + 3y - \lambda(x + y - 4)$

Computing the gradient of the Lagrangian function and setting it equal to zero, we get:

$$\partial L/\partial x = 2 - \lambda = 0$$
$$\partial L/\partial y = 3 - \lambda = 0$$
$$\partial L/\partial \lambda = x + y - 4 = 0$$

Solving these equations simultaneously, we get:

x = 2

y = 2

λ = 2

Therefore, the maximum value of the function is:

$$f(2, 2) = 2(2) + 3(2) = 10$$

Advantages and Disadvantages

Advantages:

- The Lagrange Multiplier Technique can be used to solve complex constrained optimization problems.
- It provides a systematic approach to solving constrained optimization problems.

Disadvantages:

- The Lagrange Multiplier Technique can be computationally intensive.
- It requires the computation of the gradient of the Lagrangian function, which can be difficult for complex problems.

VECTORS

Euclidean spaces are useful for modeling a wide variety of economic phenomena because n-tuples of numbers have many useful interpretations. Thus far we have emphasized their interpretation as locations, or points in n-space. For example, the point (3, 2) represents a particular location in the plane, found by going 3 units to the right and 2 units up from the origin. This is just the way we use coordinates on a map of a country to find the location of a particular city. We use coordinates to describe locations in exactly the same way in higher dimensions.

Many economic applications require us to think of n-tuples of numbers as locations. For example, we think of consumption bundles as locations in commodity space.

We can also interpret n-tuples as **displacements**. This is a useful way of thinking about vectors for doing calculus. We picture these displacements as arrows in $\mathbf{R}^n$.

The displacement (3, 2) means: move 3 units to the right and 2 units up from your current location. The tail of the arrow marks the initial location; the head marks the location after the displacement is made. In Figure 10.6, each arrow represents the displacement (3, 2), but in each case the displacement is applied to a different initial location.

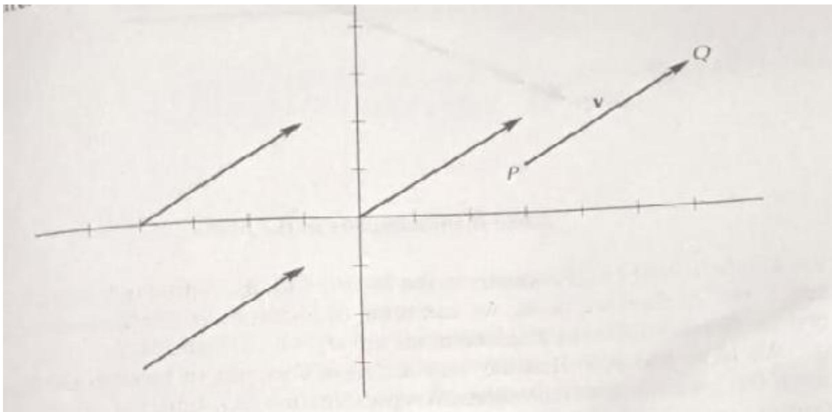


Fig:10.6 *The displacement (3,2).*

For example, the tail of the displacement labeled **v** in Figure 10.6 is at the location (3, 1), and the head is at (6, 3). We will sometimes write PQ for the displacement whose tail is at the point P and head at the point Q. Two arrows represent the same displacement if they are parallel and have the same length and direction.

For our purposes, two such arrows are equivalent; regardless of their different initial and terminal locations, they both represent the same displacement. The essential ingredients of a displacement are its magnitude and direction.

How do we assign an n-tuple to a particular arrow? We measure how far we have to move in each direction to get from the tail to the head of the arrow. For example, consider the arrow **v** in Figure 10.6. To get from the tail to the head we have to move 3 units in the x_1-direction and 2 units in the x_2-direction.

Thus **v** must represent the displacement (3, 2). More formally, if a displacement goes from the initial location (a, b) to the terminal location (c, d), then the move in the x_1-direction is c - a, since a + (c - a) = c; and the move in the x_2- direction is d - b, since b + (d - b) = d. Thus the displacement is (c - a, d - b). This method of subtracting corresponding coordinates applies to higher dimensions as well. The displacement from the point $\mathbf{p}(a_1, a_2, \ldots, a_n)$ to the point $\mathbf{q}(b_1, b_2, \ldots, b_n)$ in R^n is written

$$\overrightarrow{\mathbf{pq}} = (b_1 - a_1, b_2 - a_2, \ldots, b_n - a_n)$$

Figure 10.6 illustrates that there are many (3, 2) displacements. In any given discussion, all the displacements will usually have the same initial location (tail). Often, this initial location will naturally be 0, the origin. From this initial location,

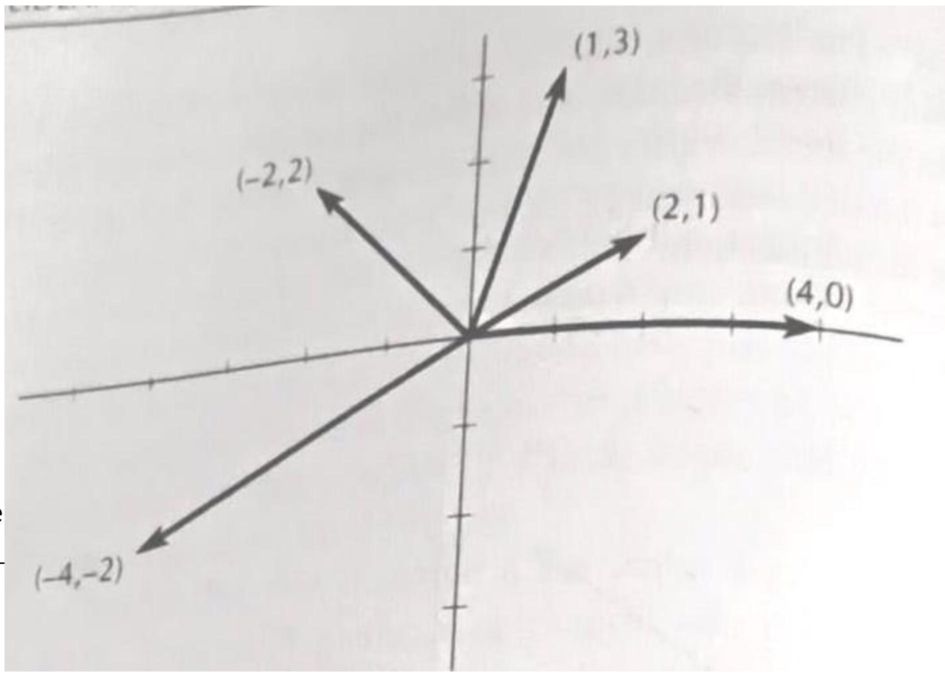


fig:10.7 Some displacements in the plane.

the displacement (3, 2) takes us to the location (3, 2). With this "canonical representation" of displacements, we can think of locations as displacements from the origin. Several different displacements are shown in Figure 10.7.

We have just seen that the very different concepts of location and displacement have a common mathematical representation as n-tuples of numbers. These concepts act alike mathematically, and so we give them a common name: **vectors**.

Some books distinguish between locations and displacements by writing a location as a row vector (a, b) and a displacement as a column vector$\left(\frac{a}{b}\right)$This approach is unwieldy and unnecessary. From now on we will use the word "vector" to refer to both locations and displacements. It will either be explicitly mentioned, or clear from the context, whether locations or displacements are meant in any particular discussion.

Jacobian Matrix

The Jacobian matrix is a matrix of partial derivatives of a vector-valued function. It is used to describe the behavior of a function near a point.

Definition

The Jacobian matrix of a function f: $\mathbb{R}^n \rightarrow \mathbb{R}^m$ is defined as:

$J = [\partial f_1/\partial x_1 \ \ \partial f_1/\partial x_2 \ \ ... \ \ \partial f_1/\partial x_n]$

$[\partial f_2/\partial x_1 \ \ \partial f_2/\partial x_2 \ \ ... \ \ \partial f_2/\partial x_n]$

[..]

$[\partial f_m/\partial x_1 \quad \partial f_m/\partial x_2 \ldots\ldots \partial f_m/\partial x_n]$

Application

The Jacobian matrix is used in:

1. Linearization: The Jacobian matrix is used to linearize a function near a point.
2. Stability Analysis: The Jacobian matrix is used to analyze the stability of a system.
3. Optimization: The Jacobian matrix is used in optimization algorithms, such as the Newton-Raphson method.

Hessian Matrix

The Hessian matrix is a matrix of second partial derivatives of a scalar-valued function. It is used to describe the curvature of a function.

Definition

The Hessian matrix of a function f: $\mathbb{R}^n \to \mathbb{R}$ is defined as:

$H = [\partial^2 f/\partial x_1^2 \quad \partial^2 f/\partial x_1\partial x_2 \ldots\ldots \partial^2 f/\partial x_1\partial x_n]$

$[\partial^2 f/\partial x_2\partial x_1 \quad \partial^2 f/\partial x_2^2 \ldots\ldots \partial^2 f/\partial x_2\partial x_n]$

[... ... .]

$[\partial^2 f/\partial x_n\partial x_1 \quad \partial^2 f/\partial x_n\partial x_2 \ldots\ldots \partial^2 f/\partial x_n^2]$

Application

The Hessian matrix is used in:

1. Optimization: The Hessian matrix is used in optimization algorithms, such as the Newton-Raphson method.
2. Stability Analysis: The Hessian matrix is used to analyze the stability of a

system.

3. Economics: The Hessian matrix is used in economics to analyze the behavior of economic systems.

Application of Jacobian and Hessian Matrices

The Jacobian and Hessian matrices are used in utility maximization to:

1. Find the optimal consumption bundle: The Jacobian matrix is used to find the optimal consumption bundle that maximizes utility.
2. Analyze the behavior of the utility function: The Hessian matrix is used to analyze the behavior of the utility function and determine the optimal consumption bundle.

Utility Maximization

Definition:

Utility maximization is the process by which consumers make choices to achieve the highest level of satisfaction (utility) given their budget constraints.

Key Assumptions:

Consumers are rational and aim to maximize their satisfaction.

They have a budget constraint that limits their choices.

Preferences are complete, transitive, and continuous (as per utility theory).

The law of diminishing marginal utility applies–each additional unit consumed provides less additional satisfaction.

Mathematical Representation:

A consumer chooses quantities of goods to maximize their utility function:

$U(x_1, x_2, ..., x_n)$

$P_1 x_1 + P_2 x_2 + ... + P_n x_n \leq M$

= price of good

= quantity of good

= consumer's income

Optimality Condition (Utility Maximization Rule):

At equilibrium, the Marginal Utility per Dollar Spent is equal across all goods:

$$\frac{MU_1}{P_1} = \frac{MU_2}{P_2} = ... = \frac{MU_n}{P_n}$$

Graphical Representation:

Indifference Curves: Show different combinations of two goods providing the same level of utility.

Budget Line:

Represents all combinations of goods the consumer can afford. Optimal Choice: The point where the highest indifference curve is tangent to the budget line.

2. Profit Maximization

Definition:

Profit maximization is the goal of firms to achieve the highest possible difference between total revenue (TR) and total cost (TC).

$$\text{Profit} (\pi) = TR - TC$$

Key Assumptions:

Firms aim to maximize profit.

They operate under certain market structures (perfect competition, monopoly, oligopoly, etc.).

They have a production function relating inputs to outputs.

Methods of Profit Maximization:

1. Total Revenue - Total Cost Approach

A firm maximizes profit where the difference is largest.

2. Marginal Revenue - Marginal Cost Approach

A firm maximizes profit where:

MR = MC

= Marginal Revenue (change in TR from selling one more unit)

= Marginal Cost (change in TC from producing one more unit)

Market Structures and Profit Maximization:

Graphical Representation:

The profit-maximizing output is where MR = MC.

If MR > MC, the firm should increase output.

If MR < MC, the firm should reduce output.

The shutdown point occurs where Price = AVC (Average Variable Cost).

3. Cost Minimization

Definition:

Cost minimization is the process by which firms choose the optimal combination of inputs to produce a given level of output at the lowest possible cost.

Key Assumptions:

Firms aim to minimize costs for a given level of output.

They use two or more inputs (e.g., labor and capital).

The production function determines how inputs produce output.

The law of diminishing marginal returns applies.

Mathematical Representation:

A firm minimizes cost subject to the production function

$C = wL + rK$

$Q = f(L, K)$

= wage rate of labor ()

= rental price of capital ()

= output level

Optimality Condition (Cost Minimization Rule):

At equilibrium, the Marginal Rate of Technical Substitution (MRTS) equals the input price ratio:

$$\frac{MP_L}{MP_K} = \frac{w}{r}$$

= Marginal Product of Labor

= Marginal Product of Capital

Graphical Representation:

Isoquants: Curves representing different combinations of labor and capital that produce the same output.

Isocost Line: All combinations of inputs that cost the same.

Optimal Input Combination: The point where the isoquant is tangent to the isocost line.

Conclusion

Utility maximization explains consumer behavior and how they allocate income among goods. Profit maximization explains firm behavior and how they choose output levels. Cost minimization ensures that firms use inputs efficiently to produce goods at the lowest possible cost.

UNIT IV

Linear and Non-Linear Programming

Optimization with inequality constraints is a type of constrained optimization where the goal is to find the optimal value of an objective function, subject to limitations or restrictions on the decision variables, expressed as inequalities.

Constraints:

Inequality constraints impose conditions that the solution must satisfy, such as upper or lower bounds on variables, or relationships between them.

Feasible Region:

The set of all possible solutions that satisfy all constraints (both equality and inequality) is called the feasible region.

Examples:

- Production Planning: Maximizing profit while ensuring that production levels don't exceed capacity or that certain resource requirements are met.
- Investment Portfolio Optimization: Finding the best asset allocation to maximize return while adhering to risk tolerance and other investment constraints.

Methods:

Various optimization algorithms and techniques are used to solve problems with inequality constraints, including:

1. Kuhn-Tucker Conditions: These are optimality conditions that provide necessary conditions for a solution to be optimal in a constrained optimization problem.
2. Lagrange Multipliers: A method to find the optimal solution by introducing Lagrange multipliers to the objective function and constraints.
3. Interior Point Methods: Algorithms that find the optimal solution by iteratively moving towards the optimal point from within the feasible region.

Linear Programming (LP)

A technique used to optimize (maximize or minimize) a linear objective function, subject to linear constraints.

Characteristics Linear Relationships

The relationships between decision variables, constraints, and the objective function are linear.

Objective Function

A linear equation representing the goal to be optimized (e.g., maximize profit, minimize cost).

Constraints

Restrictions on decision variables, also represented as linear equations or inequalities.

Decision Variables

Variables that represent the choices or actions that can be adjusted to achieve the optimal solution.

Examples:

Resource allocation, production planning, transportation problems.

Methods:

Simplex method, graphical method.

In linear programming, the duality theorem establishes a fundamental relationship between a primal problem (a maximization or minimization problem) and its dual problem. If both problems have feasible solutions, then the optimal values of the primal and dual problems are equal, and vice versa.

Here's a more detailed explanation:

1. **Primal and Dual Problems:**

- **Primal Problem:** The original linear programming problem, which can be either a maximization or minimization problem.
- **Dual Problem:** A related linear programming problem derived from the primal problem.

2. Duality Theorem:

- **Weak Duality:**

If x is a feasible solution to the primal problem and y is a feasible solution to the dual problem, then the objective function value of the primal problem is always less than or equal to the objective function value of the dual problem (for a maximization primal problem).

- **Strong Duality:**

If both the primal and dual problems have feasible solutions, then the optimal values of the primal and dual problems are equal.

- **Complementary Slackness:**

If x* and y* are optimal solutions to the primal and dual problems, respectively, then certain relationships hold between the primal and dual variables and constraints.

3. Implications of Duality:

- **Computational Efficiency:**

  Solving the dual problem can sometimes be computationally easier than solving the primal problem.

- **Economic Interpretation:**

  The dual variables can be interpreted as shadow prices or marginal values of the resources in the primal problem.

- **Optimality Conditions:**

  The duality theorem provides conditions for determining whether a solution to a linear programming problem is optimal.

- **Solving Linear Programs:**

  The duality theorem can be used to solve linear programs by solving the dual problem instead of the primal problem, or by using both problems to find the optimal solution.

4. Example:

- **Primal Problem (Maximization):**

  Maximize Z = 3x1 + 5x2 subject to 2x1 + x2 <= 8, x1 + 2x2 <= 10, x1, x2 >= 0

- **Dual Problem (Minimization):**

  Minimize W = 8y1 + 10y2 subject to 2y1 + y2 >= 3, y1 + 2y2 >= 5, y1, y2 >= 0

Simplex method

The simplex method is a systematic, iterative approach to finding the best possible solution (maximize or minimize) for a linear programming problem, which involves optimizing a linear objective function subject to linear constrain

Explanation of Simplex Method

Introduction

The Simplex method is an approach to solving linear programming models by hand using slack variables, tableaus, and pivot variables as a means to finding the optimal solution of an optimization problem. A linear program is a method of achieving the best outcome given a maximum or minimum equation with linear constraints. Most linear programs can be solved using an online solver such as MatLab, but the Simplex method is a technique for solving linear programs by hand. To solve a linear programming model using the Simplex method the following steps are necessary:

- Standard form
- Introducing slack variables
- Creating the tableau
- Pivot variables
- Creating a new tableau
- Checking for optimality
- Identify optimal values

This document breaks down the Simplex method into the above steps and follows the example linear programming model shown below throughout the entire

$$Minimize : -z = -8x_1 - 10x_2 - 7x_3$$
$$s.t. : x_1 + 3x_2 + 2x_3 \leq 10$$
$$-x_1 - 5x_2 - x_3 \geq -8$$
$$x_1, x_2, x_3 \geq 0$$

document to find the optimal solution.

Step 1: Standard Form

Standard form is the baseline format for all linear programs before solving for the optimal solution and has three requirements: (1) must be a maximization problem, (2) all linear constraints must be in a less-than-or-equal-to inequality, (3) all variables are non-negative. These requirements can always be satisfied by transforming any given linear program using basic algebra and substitution. Standard form is necessary because it creates an ideal starting point for solving the Simplex method as efficiently as possible as well as other methods of solving optimization problems.

To transform a minimization linear program model into a maximization linear

$$-1 \times (-z = -8x_1 - 10x_2 - 7x_3)$$
$$z = 8x_1 + 10x_2 + 7x_3$$
$$Maximize : z = 8x_1 + 10x_2 + 7x_3$$

program model, simply multiply both the left and the right sides of the objective function by -1.

Transforming linear constraints from a greater-than-or-equal-to inequality to a less-than-or-equal-to inequality can be done similarly as what was done to the objective function. By multiplying by -1 on both sides, the inequality can be changed to less-than-or-equal-to.

$$-1 \times (-x_1 - 5x_2 - x_3 \geq -8)$$
$$x_1 + 5x_2 + x_3 \leq 8$$

Once the model is in standard form, the slack variables can be added as shown in Step 2 of the Simplex method.

Step 2: Determine Slack Variables

Slack variables are additional variables that are introduced into the linear constraints of a linear program to transform them from inequality constraints to equality constraints. If the model is in standard form, the slack variables will always have a

$$x_1 + 3x_2 + 2x_3 + \mathbf{s_1} = 10$$
$$x_1 + 5x_2 + x_3 + \mathbf{s_2} = 8$$
$$x_1, x_2, x_3, \mathbf{s_1}, \mathbf{s_2} \geq 0$$

+1 coefficient. Slack variables are needed in the constraints to transform them into solvable equalities with one definite answer.

After the slack variables are introduced, the tableau can be set up to check for optimality as described in Step 3.

Step 3: Setting up the Tableau

A Simplex tableau is used to perform row operations on the linear programming model as well as to check a solution for optimality. The tableau consists of the

$$Maximize: z = 8x_1 + 10x_2 + 7x_3$$
$$s.t.: x_1 + 3x_2 + 2x_3 + s_1 = 10$$
$$x_1 + 5x_2 + x_3 + s_2 = 8$$

coefficient corresponding to the linear constraint variables and the coefficients of the objective function. In the tableau below, the bolded top row of the tableau states

what each column represents. The following two rows represent the linear constraint variable coefficients from the linear programming model, and the last row represents the objective function variable coefficients.

x1	**x2**	**x3**	**s1**	**s2**	**z**	**b**
1	3	2	1	0	0	10
1	5	1	0	1	0	8
-8	-10	-7	0	0	1	0

Once the tableau has been completed, the model can be checked for an optimal solution as shown in Step 4.

Step 4: Check Optimality

The optimal solution of a maximization linear programming model are the values assigned to the variables in the objective function to give the largest zeta value. The optimal solution would exist on the corner points of the graph of the entire model. To check optimality using the tableau, all values in the last row must contain values greater than or equal to zero. If a value is less than zero, it means that variable has not reached its optimal value. As seen in the previous tableau, three negative values exists in the bottom row indicating that this solution is not optimal. If a tableau is not optimal, the next step is to identify the pivot variable to base a new tableau on, as described in Step 5.

Step 5: Identify Pivot Variable

The pivot variable is used in row operations to identify which variable will become the unit value and is a key factor in the conversion of the unit value. The pivot variable can be identified by looking at the bottom row of the tableau and the indicator. Assuming that the solution is not optimal, pick the smallest negative value in the bottom row. One of the values lying in the column of this value will be the pivot variable. To find the indicator, divide the beta values of the linear constraints by their corresponding values from the column containing the possible pivot variable. The intersection of the row with the smallest non-negative indicator and the smallest negative value in the bottom row will become the pivot variable.

In the example shown below, -10 is the smallest negative in the last row. This will designate the x_2 column to contain the pivot variable. Solving for the indicator gives us a value of $\frac{10}{3}$ for the first constraint, and a value of $\frac{8}{5}$ for the second constraint. Due to $\frac{8}{5}$ being the smallest non-negative indicator, the pivot value will be in the second row and have a value of 5.

x1	x2	x3	s1	s2	z	b	Indicator
1	3	2	1	0	0	10	10/3
1	(5)	1	0	1	0	8	8/5
-8	-10	-7	0	0	1	0	
	↑ Smallest Value						

Now that the new pivot variable has been identified, the new tableau can be created in Step 6 to optimize the variable and find the new possible optimal solution.

Step 6: Create the New Tableau

The new tableau will be used to identify a new possible optimal solution. Now that the pivot variable has been identified in Step 5, row operations can be performed to optimize the pivot variable while keeping the rest of the tableau equivalent.

I. To optimize the pivot variable, it will need to be transformed into a unit value (value of 1). To transform the value, multiply the row containing the pivot variable by the reciprocal of the pivot value. In the example below, the pivot variable is originally 5, so multiply the entire row by $\frac{1}{5}$

x1	x2	x3	s1	s2	z	b	
1/5	(1)	1/5	0	1/5	0	8/5	← Pivot row

II. After the unit value has been determined, the other values in the column containing the unit value will become zero. This is because the x_2 in the second constraint is being optimized, which requires x_2 in the other equations to be zero.

x1	x2	x3	s1	s2	z	b
	0					
1/5	(1)	1/5	0	1/5	0	8/5
	0					

← Pivot row

↑ Pivot Column

III. In order to keep the tableau equivalent, the other variables not contained in the pivot column or pivot row must be calculated by using the new pivot values. For each new value, multiply the negative of the value in the old pivot column by the value in the new pivot row that corresponds to the value being calculated. Then add this to the old value from the old tableau to produce the new value for the new tableau. This step can be condensed into the equation on the next page:

New tableau value = (Negative value in old tableau pivot column) x (value in new tableau pivot row) + (Old tableau value)

Old Tableau:

x1	x2	x3	s1	s2	z	b
1	3	2	1	0	0	10
1	(5)	1	0	1	0	8
-8	-10	-7	0	0	1	0

↑ Old pivot column

New Tableau:

x1	x2	x3	s1	s2	z	b	
2/5	0	7/5	1	-3/5	0	26/5	
1/5	1	1/5	0	1/5	0	8/5	← New pivot row
-6	0	-5	0	2	1	16	

Numerical examples are provided below to help explain this concept a little better.

Numerical examples:

I. To find the s_2 value in row 1:

New tableau value = (Negative value in old tableau pivot column) * (value in new tableau pivot row) + (Old tableau value)

New tableau value = $(-3) * (\frac{1}{5}) + 0 = -\frac{3}{5}$

II. To find the x_1 variable in row 3:

New tableau value = (Negative value in old tableau pivot column) * (value in new tableau pivot row) + (Old tableau value)

New value = $(10) * (\frac{1}{5}) + -8 = -6$

Once the new tableau has been completed, the model can be checked for an optimal solution.

Step 7: Check Optimality

As explained in Step 4, the optimal solution of a maximization linear programming model are the values assigned to the variables in the objective function to give the largest zeta value. Optimality will need to be checked after each new tableau to see if a new pivot variable needs to be identified. A solution is considered optimal if all values in the bottom row are greater than or equal to zero. If all values are greater than or equal to zero, the solution is considered optimal and Steps 8 through 11 can be ignored. If negative values exist, the solution is still not optimal and a new pivot point will need to be determined which is demonstrated in Step 8.

Step 8: Identify New Pivot Variable

If the solution has been identified as not optimal, a new pivot variable will need to be determined. The pivot variable was introduced in Step 5 and is used in row operations to identify which variable will become the unit value and is a key factor in the conversion of the unit value. The pivot variable can be identified by the intersection of the row with the smallest non-negative indicator and the smallest negative value in the bottom row.

x1	x2	x3	s1	s2	z	b	Indicator
2/5	0	7/5	1	−3/5	0	26/5	(26/5) / (2/5) = 13
(1/5)	1	1	0	1/5	0	8/5	(8/5) / (1/5) = 8
-6	0	-5	0	2	1	0	

↑ Smallest Value

With the new pivot variable identified, the new tableau can be created in Step 9.

Step 9: Create New Tableau

After the new pivot variable has been identified, a new tableau will need to be created. Introduced in Step 6, the tableau is used to optimize the pivot variable while keeping the rest of the tableau equivalent.

I. Make the pivot variable 1 by multiplying the row containing the pivot variable by the reciprocal of the pivot value. In the tableau below, the pivot value was $\frac{1}{5}$, so everything is multiplied by 5.

x1	x2	x3	s1	s2	z	b
(1)	5	1	0	1	0	8

II. Next, make the other values in the column of the pivot variable zero. This is done by taking the negative of the old value in the pivot column and multiplying it by the new value in the pivot row. That value is then added to the old value that is being replaced.

x1	x2	x3	s1	s2	z	b
0	-2	1	1	-1	0	2
①	5	1	0	1	0	8
0	30	1	0	8	1	64

Step 10: Check Optimality

Using the new tableau, check for optimality. Explained in Step 4, an optimal solution appears when all values in the bottom row are greater than or equal to zero. If all values are greater than or equal to zero, skip to Step 12 because optimality has been reached. If negative values still exist, repeat steps 8 and 9 until an optimal solution is obtained.

Step 11: Identify Optimal Values

Once the tableau is proven optimal the optimal values can be identified. These can be found by distinguishing the basic and non-basic variables. A basic variable can be classified to have a single 1 value in its column and the rest be all zeros. If a variable does not meet this criteria, it is considered non-basic. If a variable is non- basic it means the optimal solution of that variable is zero. If a variable is basic, the row that contains the 1 value will correspond to the beta value. The beta value will represent the optimal solution for the given variable.

x1	x2	x3	s1	s2	z	b
0	-2	1	1	-1	0	2
1	5	1	0	1	0	8
0	30	1	0	8	1	64

Basic variables: x_1, s_1, z

Non-basic variables: x_2, x_3, s_2

For the variable x_1, the 1 is found in the second row. This shows that the optimal x_1 value is found in the second row of the beta values, which is 8.

Variable s_1 has a 1 value in the first row, showing the optimal value to be 2 from the beta column. Due to s_1 being a slack variable, it is not actually included in the optimal solution since the variable is not contained in the objective function.

The zeta variable has a 1 in the last row. This shows that the maximum objective value will be 64 from the beta column.

The final solution shows each of the variables having values of:

x_1	= 8	s_1	= 2
x_2	= 0	s_2	= 0
x_3	= 0	z	= 64

The maximum optimal value is 64 and found at (8, 0, 0) of the objective function.

Conclusion

The Simplex method is an approach for determining the optimal value of a linear program by hand. The method produces an optimal solution to satisfy the given constraints and produce a maximum zeta value. To use the Simplex method, a given linear programming model needs to be in standard form, where slack variables can then be introduced. Using the tableau and pivot variables, an optimal solution can be reached. From the example worked throughout this document, it can be determined that the optimal objective value is 64 and can be found when $x_1=8$, $x_2=0$, and $x_3=0$.

Simplex Method for Solving Linear Programming Problem (LPP)

Problem Statement:

Maximize

$Z=3x_1+5x_2$

Subject to constraints:

$x_1+2x_2\leq 4$

$3x_1+2x_2\leq 6$

$x_1,x_2\geq 0$

Step 1: Convert to Standard Form

Introduce **slack variables** s_1 and s_2 to convert inequalities into equalities:

$x_1+2x_2+s_1=43x_1+2x_2+s_2=6$

Objective function remains:

$Z-3x_1-5x_2=0$

Now,the variables are:

Decision Variables: x_1,x_2

Slack Variables: s_1,s_2

Initial **Basic Feasible Solution (BFS)**:

$x_1=0,x_2=0,s_1=4,s_2=6$

Step 2: Construct the Initial Simplex Tableau

Basis	x_1	x_2	s_1	s_2	RHS
s_1	1	2	1	0	4
s_2	3	2	0	1	6
Z	-3	-5	0	0	0

Step 3: Identify the Pivot Column (Entering Variable)

- The **most negative value in the last row (Z-row)** is -5 (for x_2), so **x_2 enters the basis.**

Step 4: Identify the Pivot Row (Leaving Variable)

- Compute the **ratio** (RHS / Pivot Column) for positive values only:
  - **Row 1:**4/2=24 / 2 = 2
  - **Row 2:**6/2=36 / 2 = 3
- The smallest ratio is **2**, so **s_1 leaves the basis.**

Pivot Element = 2 (at row 1, column 2)

Step 5: Perform Row Operations

Make the pivot element 1 (Divide row 1 by 2):

$R_1 \rightarrow R_1/2$

Update the remaining rows to make other elements in the pivot column zero:

$R_2 \rightarrow R_2 - (2 \times R_1)$

$RZ \rightarrow RZ + (5 \times R_1)$

After performing row operations, the **new tableau** is:

Basis	**x_1**	**x_2**	**s_1**	**s_2**	**RHS**
x_2	0.5	1	0.5	0	2
s_2	2	0	-1	1	2
Z	-0.5	0	2.5	0	10

Step 6: Check for Optimality

Since all values in the Z-row (excluding RHS) are **non-negative**, the solution is **optimal**.

Step 7: Interpret the Solution

- **Optimal solution:** $x_1=0$, $x_2=2$
- **Maximum value of Z:**

  Z=3(0)+5(2)=10Z = 3(0) + 5(2) = 10

Final Answer:

Optimal Solution: $x_1=0, x_2=2, Z_{max}=10$

Graphical Method:

This method is suitable for problems with only two decision variables (e.g., x and y) because you can plot the constraints and objective function on a 2D graph (Cartesian coordinate space).

Steps Involved:

1. **Formulate the Problem:** Define the objective function (what you want to maximize or minimize) and the constraints (limitations or restrictions) as linear equations or inequalities.

2. **Plot the Constraints:** Convert each constraint inequality into a line on the graph and identify the feasible region (the area that satisfies all constraints).
3. **Identify Feasible Region:** The feasible region is the area on the graph that represents all possible solutions that satisfy the constraints.
4. **Determine Optimal Solution:** Plot the objective function (as a line) and move it parallel to itself until it touches the farthest point (for maximization) or closest point (for minimization) within the feasible region. This point represents the optimal solution.
5. **Corner Point Method:** The optimal solution is usually found at one of the corner points (vertices) of the feasible region.

Let us find the feasible solution for the problem of a decorative item dealer whose LPP is to maximize profit function.

$Z = 50x + 18y$ (1)

Subject to the constraints

$2x + y \leq 100$

$x + y \leq 80$

$x \geq 0, \quad y \geq 0$

Step 1: Since $x \geq 0$, y□0, we consider only the first quadrant of the xy- plane

Step 2: We draw straight lines for the equation

2x+ y = 100 (2)

x + y = 80

To determine two points on the straight line 2x + y = 100

Put y = 0, 2x = 100

x = 50

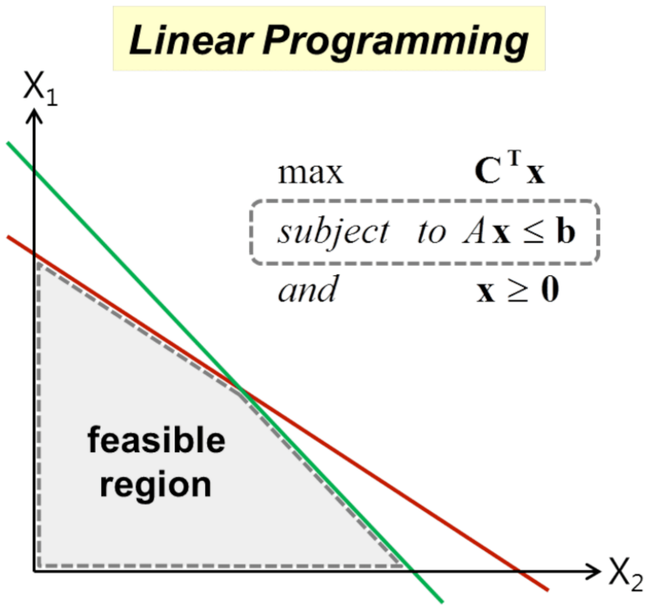
(50, 0) is a point on the line (2)

put x = 0 in (2), y =100

(0, 100) is the other point on the line (2)

Plotting these two points on the graph paper draw the line which represent the line

2x + y =100.



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n the graph, the corners of the feasible region are

O (0, 0), A (0, 80), B(20, 60), C(50, 0)

At (0, 0) Z = 0

At (0, 80) Z = 50 (0) + 18(80)

= 1440

At (20, 60), Z = 50 (20) +18 (60)

= 1000 + 1080 = Rs.2080

At (50, 0) Z = 50 (50)+ 18 (0)

= 2500.

Since our object is to maximize Z and Z has maximum at (50, 0) the optimal solution is x = 50 and y = 0.

The optimal value is 2500.

Problem

Maximize

$Z=4x_1+6x_2$

Subject to constraints:

$2x_1+x_2\leq10$

$x_1+3x_2\leq15$

$x_1,x_2\geq0$

Step 1: Convert Inequalities to Equations

To plot the feasible region, convert the inequalities into equalities:

1. **Equation for Constraint 1:**

   $2x_1+x_2=10$

   - If $x_1=0$, then $x_2=10$ (Point A: (0,10))
   - If $x_2=0$, then $x_1=5$ (Point B: (5,0))

2. **Equation for Constraint 2:**

   $x_1+3x_2=15$

   - If $x_1=0$, then $x_2=5$ (Point C: (0,5))
   - If $x_2=0$, then $x_1=15$ (Point D: (15,0))

Step 2: Identify Feasible Region

- The feasible region is bounded by the intersection of the constraint lines and is within the **first quadrant** (since $x_1,x_2\geq0$).
- The feasible region is **shaded** in the graph.

Step 3: Find the Corner Points

The feasible region's **corner points** are:

1. (0,10)

2. (5,0)

Intersection of the two constraint lines (solving equations simultaneously):

Solve:

$2x_1+x_2=10$, $x_1+3x_2=15$

Multiply the second equation by 2:

$2x_1+6x_2=30$

Subtract the first equation:

$(2x_1+6x_2)-(2x_1+x_2)=30-10$

$5x_2=20 \Rightarrow x_2=45$

Substitute $x_2=4$ into $2x_1+x_2=10$

$2x_1+4=10 \Rightarrow 2x_1=6 \Rightarrow x_1=3$

Intersection point: (3,4)

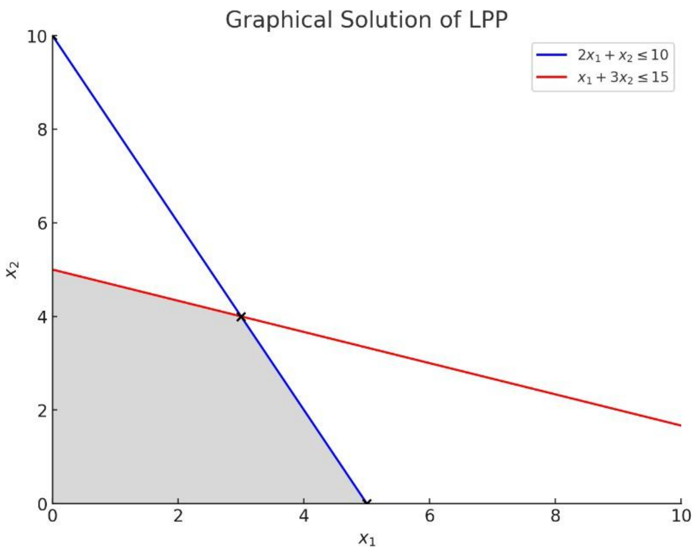
Step 4: Compute Z at Each Corner Point

Point	**x_1**	**x_2**	**$Z=4x_1+6x_2$**
(0,10)	0	10	4(0)+6(10)=60
(5,0)	5	0	4(5)+6(0)=20

Point	x_1	x_2	$Z=4x_1+6x_2$
(3,4)	3	4	4(3)+6(4)=12+24=36

Step 5: Identify Optimal Solution

- The **maximum** value of Z occurs at (0,10) with Z=60
- **Optimal solution:** $x_1=0$, $x_2=10$, $Z_{max}=60$.



Final Answer:

Optimal Solution: $x_1=0, x_2=10, Z_{max}=60$

Nonlinear Programming (NLP):

A technique used to optimize a nonlinear objective function, subject to nonlinear constraints.

Characteristics

Nonlinear Relationships: The relationships between decision variables, constraints, and the objective function can be nonlinear, represented by curves or surfaces.

Objective Function

A nonlinear equation representing the goal to be optimized.

Constraints

Restrictions on decision variables, which can also be nonlinear.

Decision Variables

Variables that represent the choices or actions that can be adjusted to achieve the optimal solution.

Examples:

Portfolio optimization, optimal control, and nonlinear regression.

Methods:

Gradient descent, Newton's method, and other iterative optimization algorithms.

Karush-Kuhn-Tucker

The Karush-Kuhn-Tucker (KKT) conditions are a set of necessary conditions for a solution to be optimal in a constrained nonlinear programming problem, generalizing the method of Lagrange multipliers to include inequality constraints.

Here's a breakdown of the KKT conditions:

- **Problem Setup:**

  Consider a constrained optimization problem:

  - Minimize $f(x)$ subject to $g(x) \leq 0$ (inequality constraints) and $h(x) = 0$ (equality constraints).

- **Lagrange Function:**

  Formulate the Lagrangian function: $L(x, \lambda, \mu) = f(x) + \lambda g(x) + \mu h(x)$.

  - Where:
    - x is the vector of decision variables.
    - λ and μ are Lagrange multipliers (dual variables) associated with the inequality and equality constraints, respectively.

- **KKT Conditions:**

  For a solution x^* to be optimal, the following conditions must hold:

- **Stationarity:** $\nabla L(x^*, \lambda^*, \mu^*) = 0$ (The gradient of the Lagrangian with respect to x is zero). This implies that the optimal point is a stationary point of the Lagrangian.
- **Primal Feasibility:** $g(x^*) \le 0$ and $h(x^*) = 0$ (The original constraints must be satisfied at the optimal point).
- **Dual Feasibility:** $\lambda^* \ge 0$ (Lagrange multipliers associated with inequality constraints must be non-negative).
- **Complementary Slackness:** $\lambda g(x) = 0$ (The product of the Lagrange multiplier and the corresponding constraint function must be zero). This means that either the Lagrange multiplier is zero, or the constraint is active (i.e., $g(x^*) = 0$).

- **Significance:**
  - The KKT conditions provide necessary conditions for optimality, meaning that if a solution is optimal, it must satisfy these conditions.
  - In some cases (e.g., convex problems), the KKT conditions are also sufficient, meaning that if they are satisfied, the solution is guaranteed to be optimal.
  - The KKT conditions are crucial for developing algorithms and understanding the behavior of constrained optimization problems.

Kuhn-Tucker Conditions in Nonlinear Programming: Solved Problems

Problem 1: Minimization with Equality Constraint

Problem Statement: $\min f(x_1,x_2)=x_1^2+x_2^2$

Subject to: $x_1+x_2-2=0$

Solution:

1. **Define Lagrangian:** $L(x_1,x_2,\mu)=x_1^2+x_2^2+\mu(x_1+x_2-2)$
2. **Compute gradients:** $\partial L/\partial x1=2x_1+\mu=0 \Rightarrow \mu=-2x_1$

   $\partial L/\partial x2=2x_2+\mu=0$

   $\Rightarrow \mu=-2x_2$

   Equating: $-2x_1=-2x_2 \Rightarrow x_1=x_2$.

3. **Satisfy the constraint:**

   $x_1+x_2=2$

   $\Rightarrow 2x_1=2$

   $\Rightarrow x_1^*=x_2^*=1$

   Optimal solution: $x_1^*=1, x_2^*=1$, $f(x_1^*,x_2^*)=2$.

Problem 2: Maximization with Multiple Constraints

$\max f(x_1,x_2)=x_1x_2$

Subject to: $x_1+2x_2-4\leq 0$

$x_1,x_2\geq 0$

Solution:

1. **Lagrangian function:** $L(x_1,x_2,\lambda)=x_1x_2+\lambda(4-x_1-2x_2)$

Compute gradients: $\partial L/\partial x_1=x_2-\lambda=0\Rightarrow\lambda=x_2$ $\partial L/\partial x_2=x_1-2\lambda=0\Rightarrow 2\lambda=x_1$

Find values using constraint $x_1+2x_2=4$

$2x_2+2x_2=4$

$\Rightarrow 4x_2=4$

$\Rightarrow x2=1$

$x_1=2(1)=2$

Optimal solution: $x_1^*=2, x_2^*=1$, $f(x_1^*,x_2^*)=2$.

Problem 3: Quadratic Programming

$\min f(x_1,x_2)=x_1^2+4x_2^2$

Subject to: $x_1+x_2-3\leq 0$

$x_1,x_2\geq 0$

Solution:

Lagrangian function: $L(x_1,x_2,\lambda)=x_1^2+4x_2^2+\lambda(3-x_1-x_2)$

Compute gradients:

$2x_1-\lambda=0$

$\Rightarrow \lambda = 2x_1^2$

$8x_2 - \lambda = 0$

$\Rightarrow \lambda = 8x_2$

Solve for x_1 and x_2 using $x_1 + x_2 = 3$

$2x_1 = 8x_2$

$\Rightarrow x_1 = 4x_2$

$4x_2 + x_2 = 3$

$\Rightarrow 5x_2 = 3$

$\Rightarrow x_2 = 0.6, x_1 = 2.4$

Optimal solution:

$x_1^* = 2.4, x_2^* = 0.6,\ f(x_1^*, x_2^*) = 5.76$

UNIT V

ECONOMICS DYNAMICS

Differential Equations

Definition

A differential equation is an equation that involves an unknown function and its derivatives. It expresses a relationship between a function and its rates of change.

Types of Differential Equations

A. Based on Order

The order of a differential equation is the highest derivative present in the equation.

1. First-Order Differential Equation

Contains only the first derivative (dy/dx).

Example: / frac{dy}{dx} + y = x

2. Second-Order Differential Equation

Involves the second derivative (d^2y/dx^2).

Example: \frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 0

3. Higher-Order Differential Equation

Contains derivatives of order three or higher.

Example: $\frac{d^3y}{dx^3} - 4\frac{d^2y}{dx^2} + 6\frac{dy}{dx} - 4y = 0$

2. Based on Linearity

1. Linear Differential Equation

The dependent variable and its derivatives appear in a linear form (no powers, products, or trigonometric functions).

Example: $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = x$

2. Nonlinear Differential Equation

Contains non-linear terms (e.g., powers, products of derivatives).

Example: $\left(\frac{dy}{dx}\right)^2 + y^2 = x$

C. Based on Nature of Coefficients

1. Constant-Coefficient Differential Equation

The coefficients of the derivatives are constants.

Example: $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0$

2. Variable-Coefficient Differential Equation

The coefficients involve the independent variable.

Example: $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$

D. Based on Homogeneity

1. Homogeneous Differential Equation

If all terms involve the dependent variable or its derivatives.

Example: $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0$

2. Non-Homogeneous Differential Equation

If a term is independent of the dependent variable or its derivatives.

Example: $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = e^x$

E. Based on Partial vs. Ordinary

1. Ordinary Differential Equation (ODE)

Involves derivatives with respect to only one independent variable.

Example: $\frac{d^2y}{dx^2} + 4y = 0$

2. Partial Differential Equation (PDE)

Involves partial derivatives with respect to multiple independent variables.

Example: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

F. Based on Exactness

1. Exact Differential Equation

Can be written in the form of an exact differential (i.e., an integrable total derivative).

Example: $(2xy + 3)dx + (x^2 + 4y)dy = 0$

2. Non-Exact Differential Equation

Cannot be written directly as an exact differential but may be solvable using an integrating factor.

G. Based on Solution Behavior

1. Initial Value Problem (IVP)

A differential equation with given initial conditions.

Example: $\frac{dy}{dx} + 2y = e^x, \quad y(0) = 1$

2. Boundary Value Problem (BVP)

A differential equation with conditions specified at different points.

Example:

$$\frac{d^2y}{dx^2} = -y, \quad y(0) = 0, \quad y(\pi) = 1$$

Applications of differential equations:

Physics and Engineering

1. Motion of Objects: Differential equations are used to model the motion of objects, including the acceleration, velocity, and position of an object.

2. Forced Vibrations: Differential equations are used to model the motion of objects that are subjected to external forces, such as springs and masses.

3. Electric Circuits: Differential equations are used to model the behavior of electric circuits, including the flow of current and the voltage across components.

4. Heat Transfer: Differential equations are used to model the transfer of heat between objects, including conduction, convection, and radiation.

Biology and Medicine

1. Population Growth: Differential equations are used to model the growth of populations, including the spread of disease and the impact of environmental factors.

2. Epidemiology: Differential equations are used to model the spread of disease, including the impact of vaccination and quarantine.

3. Pharmacokinetics: Differential equations are used to model the absorption, distribution, and elimination of drugs in the body.

4. Systems Biology: Differential equations are used to model the behavior of complex biological systems, including gene regulation and metabolic pathways.

Economics and Finance

1. Economic Growth: Differential equations are used to model the growth of economies, including the impact of investment, savings, and government policy.

2. Financial Markets: Differential equations are used to model the behavior of financial markets, including the pricing of stocks, bonds, and options.

3. Resource Management: Differential equations are used to model the management of natural resources, including the impact of harvesting and conservation.

4. Macroeconomics: Differential equations are used to model the behavior of macroeconomic systems, including the impact of monetary and fiscal policy.

Environmental Science and Ecology

1. Climate Modeling: Differential equations are used to model the behavior of the climate system, including the impact of greenhouse gases and aerosols.

2. Water Quality: Differential equations are used to model the behavior of water quality systems, including the impact of pollution and treatment.

3. Ecosystem Dynamics: Differential equations are used to model the behavior of ecosystems, including the impact of species interactions and environmental factors.

4. Conservation Biology: Differential equations are used to model the behavior of populations and ecosystems, including the impact of conservation efforts.

Differential Equations – homogenous and exact

1. **Homogeneous Differential Equations**

A first-order differential equation of the form:

$M(x, y)dx + N(x, y)dy = 0$

Solution Method:

Use substitution (where), leading to a separable equation.

Solve by integration.

2. Exact Differential Equations

A differential equation of the form:

$M(x, y)dx + N(x, y)dy = 0$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. Verify exactness.

Find the function such that:

$$\frac{\partial F}{\partial x} = M, \quad \frac{\partial F}{\partial y} = N.$$

3. First-Order Linear Differential Equations

General form:

$$\frac{dy}{dx} + P(x)y = Q(x).$$

Multiply by the integrating factor (IF):

$$IF = e^{\int P(x)dx}.$$

$$y \cdot IF = \int Q(x) \cdot IF \, dx + C.$$

4. Second-Order Linear Differential Equations with Constant Coefficients

General form:

$$a \frac{d^2y}{dx^2} + b \frac{dy}{dx} + cy = 0.$$

Solve the equation:

$$ar^2 + br + c = 0.$$

$$y = C_1 e^{r_1 x} + C_2 e^{r_2 x}.$$

$$y = (C_1 + C_2 x)e^{rx}.$$

$$y = e^{\alpha x} (C_1 \cos \beta x + C_2 \sin \beta x).$$

Solow's Model

Solow's Model, also known as the Solow Growth Model, is a neoclassical economic growth model developed by Robert Solow in the 1950s.

Assumptions:

1. Constant Returns to Scale: The production function exhibits constant returns to scale.

2. Diminishing Marginal Productivity: The marginal productivity of capital and labor decreases as their quantities increase.

3. Exogenous Technological Progress: Technological progress is exogenous and occurs at a constant rate.

Models-3

1. Solow's Model

The Solow Model consists of the following equations:

1. Production Function: $Y = AK^{\alpha}L^{(1-\alpha)}$

2. Capital Accumulation: $K' = sY - \delta K$

3. Labor Growth: $L' = nL$

where:

- Y is output
- K is capital
- L is labor
- A is technological progress
- α is the output elasticity of capital
- s is the savings rate
- δ is the depreciation rate
- n is the labor growth rate

Steady-State Solution:

The steady-state solution is obtained by setting K' = 0 and L' = 0.

1. Steady-State Capital: $K^* = (sA/(\delta + n))^{(1/(1-\alpha))}$
2. Steady-State Output: $Y^* = AK^{*\alpha}L^{(1-\alpha)}$

Convergence:

The Solow Model predicts that economies will converge to their steady-state values, regardless of their initial conditions.

2. Harrod Model

The Harrod Model, also known as the Harrod-Domar Model, is a Keynesian economic growth model developed by Roy Harrod in the 1930s.

Assumptions:

1. Constant Returns to Scale: The production function exhibits constant returns to scale.

2. Diminishing Marginal Productivity: The marginal productivity of capital decreases as its quantity increases.

3. Exogenous Savings Rate: The savings rate is exogenous and constant.

The Harrod Model consists of the following equations:

1. Production Function: $Y = AK$

2. Capital Accumulation: $K' = sY$

3. Labor Growth: $L' = nL$

where:

- Y is output

- K is capital

- A is technological progress

- s is the savings rate

- n is the labor growth rate

Steady-State Solution:

The steady-state solution is obtained by setting K' = 0.

1. Steady-State Capital: $K^* = sA/(n + \delta)$

2. Steady-State Output: $Y^* = AK^*$

Warranted Growth Rate:

The Harrod Model predicts that the economy will grow at a warranted growth rate, which is equal to the savings rate divided by the capital-output ratio.

3. Domar Model

The Domar Model is a Keynesian economic growth model developed by Evsey Domar in the 1940s.

Assumptions:

1. Constant Returns to Scale: The production function exhibits constant returns to scale.

2. Diminishing Marginal Productivity: The marginal productivity of capital decreases as its quantity increases.

3. Exogenous Investment Rate: The investment rate is exogenous and constant.

Model:

The Domar Model consists of the following equations:

1. Production Function: $Y = AK$

2. Capital Accumulation: $K' = I$

3. Labor Growth: $L' = nL$

where:

- Y is output
- K is capital
- A is technological progress
- I is investment
- n is the labor growth rate

Steady-State Solution:

The steady-state solution is obtained by setting $K' = 0$.

1. Steady-State Capital: $K^* = I/(n + \delta)$

2. Steady-State Output: $Y^* = AK^*$

Applications to Market Models

1. Neoclassical Growth Model: The Solow Model is used to study the long-run growth of economies.

2. Keynesian Cross Model: The Harrod Model is used to study the determination of aggregate demand and supply.

3. Investment and Growth Model: The Domar Model is used to study the relationship between investment and economic growth.

4. Endogenous Growth Model: The Solow Model is used as a foundation for endogenous growth models, which study the role of innovation and human capital in economic growth.

These models have been widely used in various fields, including:

1. Macroeconomics: To study the behavior of aggregate variables, such as GDP, inflation, and unemployment.

2. Economic Growth: To study the long-run growth of economies and the factors that influence it.

3. Development Economics

A linear difference equation with constant coefficients

Definition

A linear difference equation with constant coefficients is an equation of the form:

$y(t + 1) + a_1 y(t) = b$

where:

- $y(t)$ is the dependent variable

- t is the independent variable (usually representing time)

- a_1 is a constant coefficient

- b is a constant term

First-Order Linear Difference Equation

A first-order linear difference equation is an equation of the form:

$y(t + 1) + a_1y(t) = b$

where:

- $y(t)$ is the dependent variable
- t is the independent variable (usually representing time)
- a_1 is a constant coefficient
- b is a constant term

Solution Method

The solution to a first-order linear difference equation can be found using the following steps:

1. Homogeneous Solution: Find the solution to the homogeneous equation:

$y(t + 1) + a_1y(t) = 0$

The solution to this equation is:

$y_h(t) = c * (-a_1)^t$

where c is an arbitrary constant.

2. Particular Solution: Find a particular solution to the nonhomogeneous equation:

$y(t + 1) + a_1y(t) = b$

A particular solution can be found by assuming a solution of the form:

$y_p(t) = k$

where k is a constant.

Substituting this solution into the nonhomogeneous equation yields:

$k + a_1k = b$

Solving for k yields:

$k = b / (1 + a_1)$

3. General Solution: The general solution to the first-order linear difference equation is:

$y(t) = y_h(t) + y_p(t)$

Substituting the expressions for $y_h(t)$ and $y_p(t)$ yields:

$y(t) = c * (-a_1)^t + b / (1 + a_1)$

Stability Analysis

The stability of the solution to a first-order linear difference equation can be analyzed by examining the value of the coefficient a_1.

- If $|a_1| < 1$, the solution is stable and converges to the particular solution.

- If $|a_1| > 1$, the solution is unstable and diverges from the particular solution.

- If $|a_1| = 1$, the solution is marginally stable and may oscillate or converge to the particular solution.

Applications

First-order linear difference equations have numerous applications in various fields, including:

1. Economics: Modeling population growth, inflation, and economic systems.

2. Finance: Modeling stock prices, interest rates, and investment returns.

3. Biology: Modeling population dynamics, disease spread, and chemical reactions.

4. Engineering: Modeling control systems, signal processing, and communication networks.

linear difference equations with constant coefficients, focusing on second-order equations:

Definition

A linear difference equation with constant coefficients is an equation of the form:

$y(t + 2) + a_1y(t + 1) + a_2y(t) = b$

where:

- $y(t)$ is the dependent variable

- t is the independent variable (usually representing time)

- a_1 and a_2 are constant coefficients

- b is a constant term

Second-Order Linear Difference Equation

A second-order linear difference equation is an equation of the form:

$y(t + 2) + a_1y(t + 1) + a_2y(t) = b$

where:

- $y(t)$ is the dependent variable
- t is the independent variable (usually representing time)
- a_1 and a_2 are constant coefficients
- b is a constant term

Solution Method

The solution to a second-order linear difference equation can be found using the following steps:

1. Homogeneous Solution: Find the solution to the homogeneous equation:

$y(t + 2) + a_1y(t + 1) + a_2y(t) = 0$

The solution to this equation is:

$y_h(t) = c_1 * r_1^t + c_2 * r_2^t$

where c_1 and c_2 are arbitrary constants, and r_1 and r_2 are the roots of the characteristic equation:

$r^2 + a_1r + a_2 = 0$

2. Particular Solution: Find a particular solution to the nonhomogeneous equation:

$y(t + 2) + a_1y(t + 1) + a_2y(t) = b$

A particular solution can be found by assuming a solution of the form:

$y_p(t) = k$

where k is a constant.

Substituting this solution into the nonhomogeneous equation yields:

$k + a_1k + a_2k = b$

Solving for k yields:

$k = b / (1 + a_1 + a_2)$

3. General Solution: The general solution to the second-order linear difference equation is:

$y(t) = y_h(t) + y_p(t)$

Substituting the expressions for $y_h(t)$ and $y_p(t)$ yields:

$y(t) = c_1 * r_1\text{^}t + c_2 * r_2\text{^}t + b / (1 + a_1 + a_2)$

Characteristic Equation

The characteristic equation is a quadratic equation that determines the nature of the solution:

$r\text{^}2 + a_1r + a_2 = 0$

The roots of the characteristic equation can be real and distinct, real and repeated, or complex conjugates.

Stability Analysis

The stability of the solution to a second-order linear difference equation can be analyzed by examining the roots of the characteristic equation:

- If the roots are real and distinct, and $|r_1| < 1$ and $|r_2| < 1$, the solution is stable and converges to the particular solution.

- If the roots are real and distinct, and $|r_1| > 1$ or $|r_2| > 1$, the solution is unstable and diverges from the particular solution.

- If the roots are real and repeated, and $|r_1| < 1$, the solution is stable and converges to the particular solution.

- If the roots are complex conjugates, and $|r_1| < 1$, the solution is stable and converges to the particular solution.

Applications

Second-order linear difference equations have numerous applications in various fields, including:

1. Economics: Modeling economic systems, including inflation, unemployment, and GDP.

2. Finance: Modeling stock prices, interest rates, and investment returns.

3. Biology: Modeling population dynamics, disease spread, and chemical reactions.

4. Engineering: Modeling control systems, signal processing, and communication networks.

5. Physics: Modeling mechanical systems, electrical circuits, and quantum systems.

Samuelson accelerator model

The Samuelson accelerator model is based on the following assumptions:

1. Consumption function: Consumption (C) is a function of current income (Y), and is given by the equation $C = a + bY$, where a and b are constants.

2. Investment function: Investment (I) is a function of the change in consumption, and is given by the equation $I = c(C - C(-1))$, where c is a constant.

3. National income: National income (Y) is equal to the sum of consumption and investment, and is given by the equation $Y = C + I$.

The Accelerator

The accelerator is the ratio of the change in investment to the change in consumption. It is given by the equation:

$$\text{Accelerator} = \Delta I / \Delta C = c$$

The accelerator is a key concept in the Samuelson accelerator model, as it describes how changes in consumption lead to changes in investment.

Stability of the Model

The stability of the Samuelson accelerator model depends on the value of the accelerator (c). If c is less than 1, the model is stable, and national income will converge to a steady-state value. If c is greater than 1, the model is unstable, and national income will oscillate.

Applications of the Model

The Samuelson accelerator model has been used to study a variety of macroeconomic phenomena, including:

1. Business cycles: The model can be used to study the fluctuations in national income that occur over the business cycle.

2. Economic growth: The model can be used to study the long-run growth of national income.

3. Fiscal policy: The model can be used to study the effects of government spending and taxation on national income.

Limitations of the Model

The Samuelson accelerator model has several limitations, including:

1. Oversimplification: The model assumes that consumption and investment are the only components of national income, and that the accelerator is constant.

2. Lack of microfoundations: The model does not provide a detailed explanation of the behavior of individual consumers and firms.

3. Ignoring other macroeconomic variables: The model ignores other important macroeconomic variables, such as inflation, interest rates, and exchange rates.

Example 3:

Suppose the consumption function is $C = 100 + 0.5Y$, and the investment function is $I = 0.8(C - C(-1))$. Is the model stable or unstable?

Solution:

$c = 0.8$

Since c is less than 1, the model is stable.

Example 4:

Suppose the consumption function is C = 200 + 0.3Y, and the investment function is I = 1.2(C - C(-1)). Is the model stable or unstable?

Solution:

c = 1.2

Since c is greater than 1, the model is unstable.

Cobweb Model

The Cobweb Model is a mathematical model used in economics to explain the dynamics of supply and demand in a market. It is a simple model that assumes a linear supply and demand curve.

Assumptions of the Cobweb Model

1. Linear Supply and Demand Curve: The supply and demand curves are assumed to be linear.

2. No Lags in Supply: The supply curve is assumed to adjust immediately to changes in price.

3. No Lags in Demand: The demand curve is assumed to adjust immediately to changes in price.

4. No Government Intervention: The model assumes that there is no government intervention in the market.

The Cobweb Model Diagram

The Cobweb Model diagram consists of a supply curve and a demand curve that intersect at the equilibrium price and quantity.

How the Cobweb Model Works

1. Initial Equilibrium: The market starts at an initial equilibrium price and quantity.

2. Supply and Demand Adjustment: The supply and demand curves adjust to changes in price.

3. New Equilibrium: A new equilibrium price and quantity are established.

4. Repeat: Steps 2-3 are repeated until the market reaches a stable equilibrium.

Types of Equilibrium in the Cobweb Model

1. Stable Equilibrium: The market converges to a stable equilibrium price and quantity.

2. Unstable Equilibrium: The market diverges from an unstable equilibrium price and quantity.

3. Neutral Equilibrium: The market remains at a neutral equilibrium price and quantity.

Applications of the Cobweb Model

1. Agricultural Markets: The Cobweb Model is used to explain price fluctuations in agricultural markets.

2. Resource Markets: The model is used to explain price fluctuations in resource markets, such as oil and minerals.

3. Financial Markets: The model is used to explain price fluctuations in financial markets, such as stocks and bonds.

Limitations of the Cobweb Model

1. Oversimplification: The model assumes a simple linear supply and demand curve.

2. No Lags: The model assumes that supply and demand adjust immediately to changes in price.

3. No Government Intervention: The model assumes that there is no government intervention in the market.

Example 1:

Suppose the demand curve is Qd = 100 - 2P, and the supply curve is Qs = 2P - 20. Find the equilibrium price and quantity.

Solution:

Set Qd = Qs and solve for P:

100 - 2P = 2P - 20

4P = 120

P = 30

Substitute P into either the demand or supply equation to find Q:

Q = 100 - 2(30) = 40

Example 2:

Suppose the demand curve is Qd = 200 - 3P, and the supply curve is Qs = 3P - 30. Find the equilibrium price and quantity.

Solution:

Set Qd = Qs and solve for P:

200 - 3P = 3P - 30

6P = 230

P = 38.33

Substitute P into either the demand or supply equation to find Q:

Q = 200 - 3(38.33) = 45

Stability of the Cobweb Model

The stability of the Cobweb Model depends on the slope of the supply and demand curves. If the supply curve is steeper than the demand curve, the model is stable. If the demand curve is steeper than the supply curve, the model is unstable.

Example 3:

Suppose the demand curve is Qd = 100 - 2P, and the supply curve is Qs = 2P - 20. Is the model stable or unstable?

Solution:

The supply curve is steeper than the demand curve, so the model is stable.

Example 4:

Suppose the demand curve is Qd = 200 - P, and the supply curve is Qs = P - 20. Is the model stable or unstable?

Solution:

The demand curve is steeper than the supply curve, so the model is unstable

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